

Chapter 2 Dynamic Models

INSTRUCTOR: **DR. XIANGYU MENG**

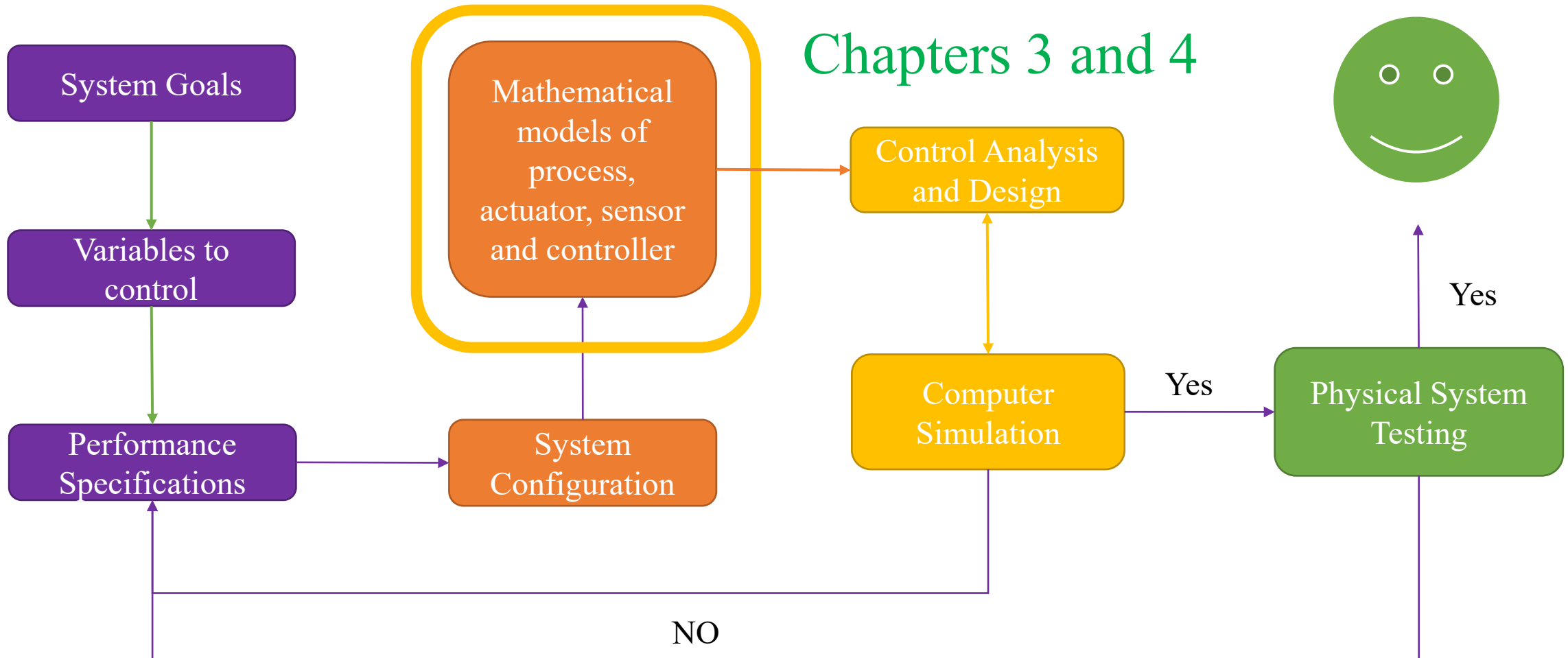
Learning Outcomes

- Recognize that **transfer functions** can describe the **dynamic behavior** of physical systems.
- Model basic **mechanical** systems using transfer functions.
- Model basic **electrical** systems using transfer functions.
- Model basic **mechatronic** systems using transfer functions.
- Understand the important role of **modeling** in the control system design process.

Control System Design Process

Chapter 2: Modeling

Chapters 3 and 4

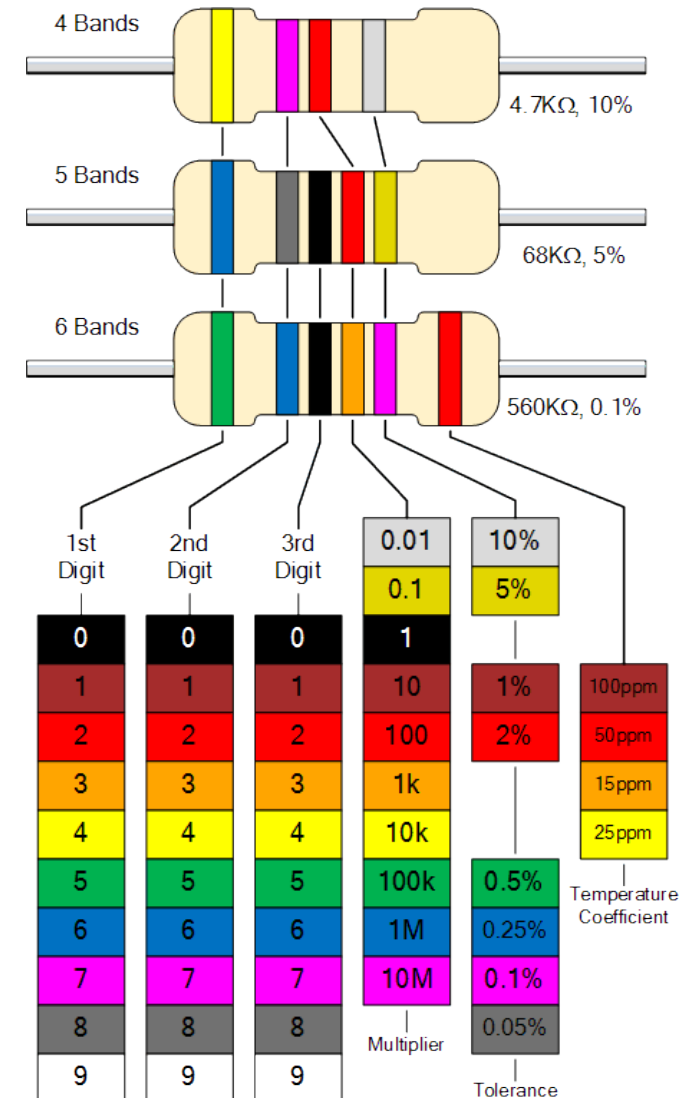


Mathematical model

- **Mathematical** models **vs.** **Physical** Systems
- Mathematical model: approximate actual system's behavior and test control design
- More **accessible**, **safer**, and **less expensive**
- Modeling is **difficult** and **expensive**
- First principle modeling **vs.** Data-driven modeling
- A model is **NOT** the same as its real-world counterpart
- Same system can be modeled in **different** forms
- A **good** model captures the system's **essential** characteristics

Example

- Physical system: a common 1-Ω 2-W carbon composition resistor
- 1-V dc power supply $p(t) = \frac{v^2(t)}{R} = 1 \text{ W}$
 - Mathematical model: current 1 A, and power
 - Physical system: current approximately 1 A, and power approximately 1 W
- 10-V dc power supply
 - Mathematical model: current 10 A, and power 100 W
 - Physical system: no current



Linear Model

- Principle of Superposition



$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$$

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Superposition $\longleftrightarrow$ System is Linear

Why Linear Model

- **NO** mathematical model of a physical system is exact.
- All physical systems are **nonlinear**.
- **Nonlinear model**: significantly more challenging to analyze
- **Linear model**: adequate for control design purpose
- **Linearization**: linearize simple nonlinearities

The Laplace Transform

- Laplace transform

$$F(s) = L\{f(t)\} \triangleq \int_0^{+\infty} f(t)e^{-st}dt$$

$s = \sigma + j\omega$



Properties of Laplace Transform

$$L\{f(t)\} = F(s)$$

- Superposition

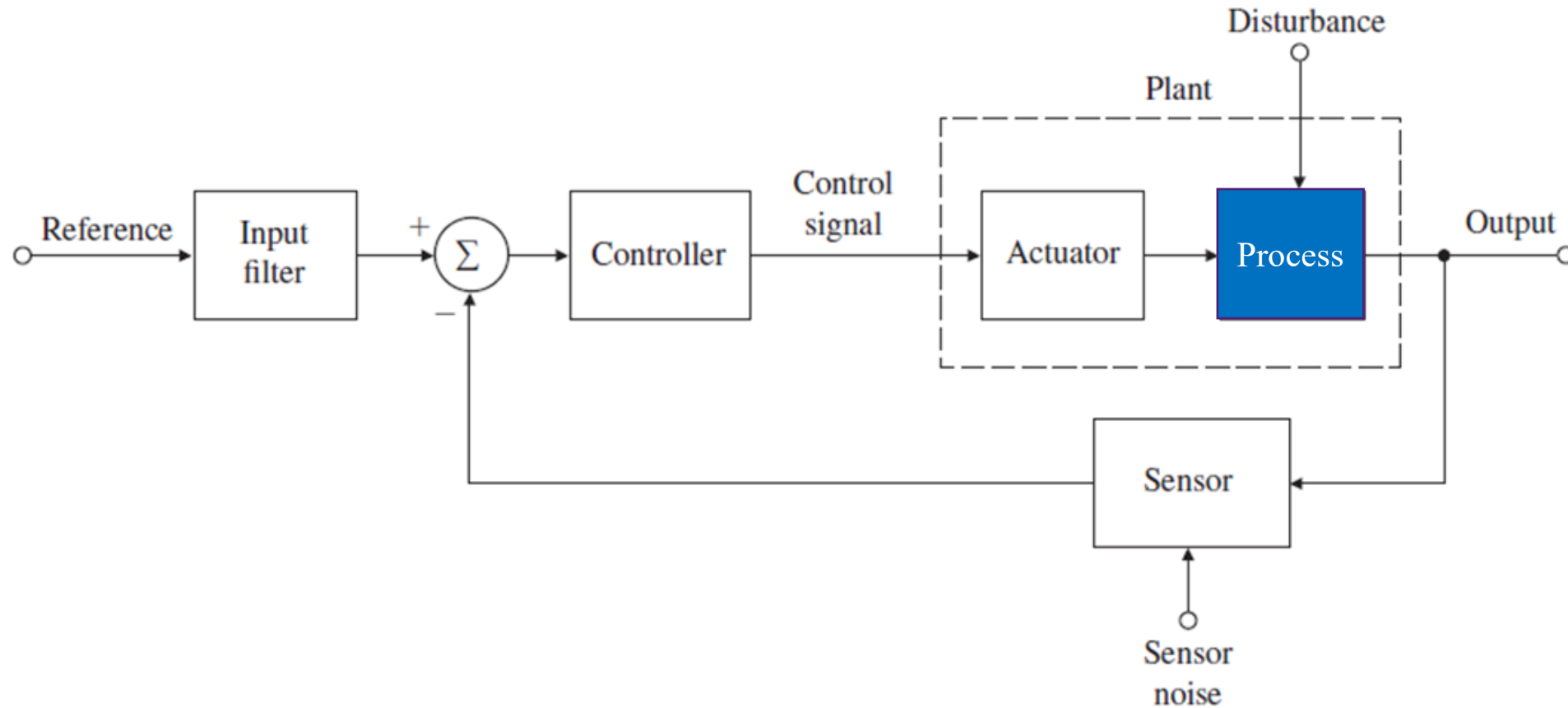
$$L\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s)$$

- Differentiation

$$L\{f^{(1)}\} = sF(s) - f(0) \qquad L\{f^{(2)}\} = s^2F(s) - sf(0) - f^{(1)}(0)$$

$$L\{f^{(m)}(t)\} = s^m F(s) - s^{m-1}f(0) - s^{m-2}f^{(1)}(0) - \dots - f^{(m-1)}(0)$$

Feedback Control Design



Dynamics of Mechanical System

Translational Motion



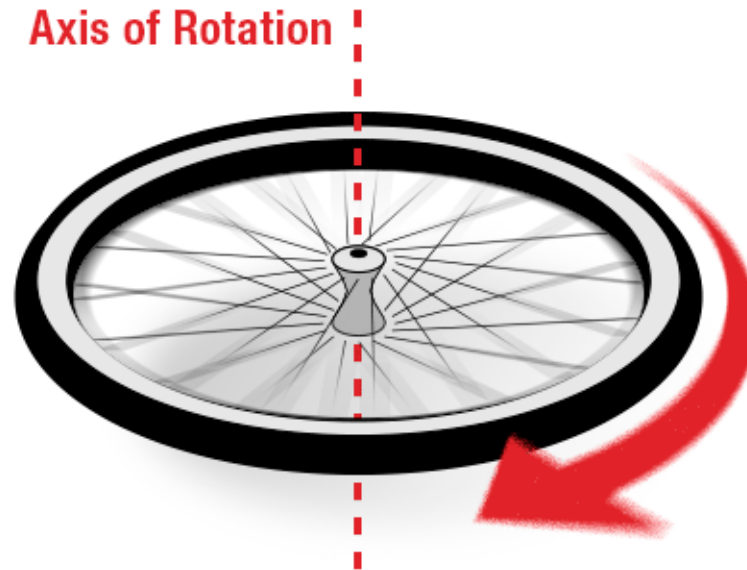
Move from one point to another

Rotational Motion

Spin around



Axis of Rotation



Combined Rotation and Translation



Translational Motion

- **Newton's law** (second law): In an **inertial reference frame**, the vector sum of the forces **$\mathbf{F}$ (N)** on an object is equal to the mass **m (kg)** of that object multiplied by the **inertial acceleration (m/sec²)** of the object

$$\mathbf{F} = m\mathbf{a}$$

- Displacement, Velocity, and Acceleration

$$v(t) = \dot{x}^{(1)}(t)$$

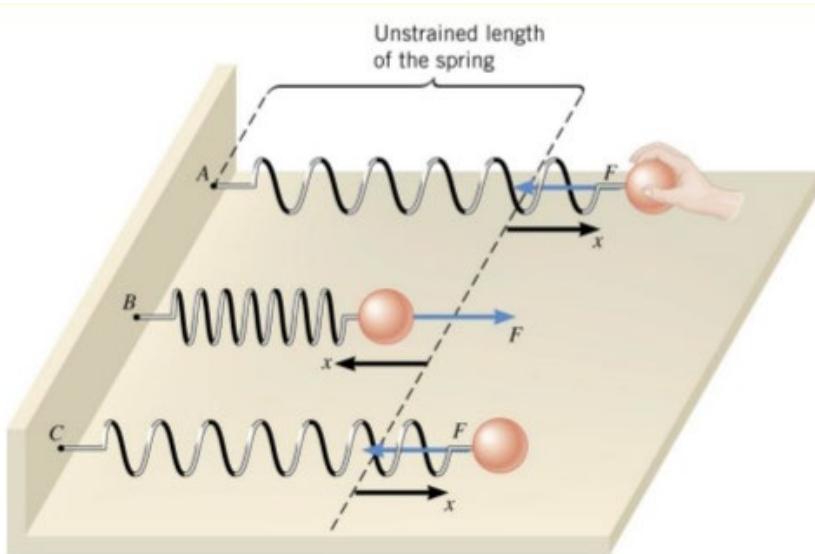
$$a(t) = \dot{v}^{(1)}(t) = \ddot{x}^{(2)}(t)$$

$$V(s) = sX(s)$$

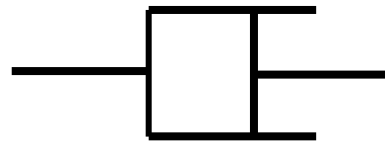
$$A(s) = sV(s) = s^2X(s)$$

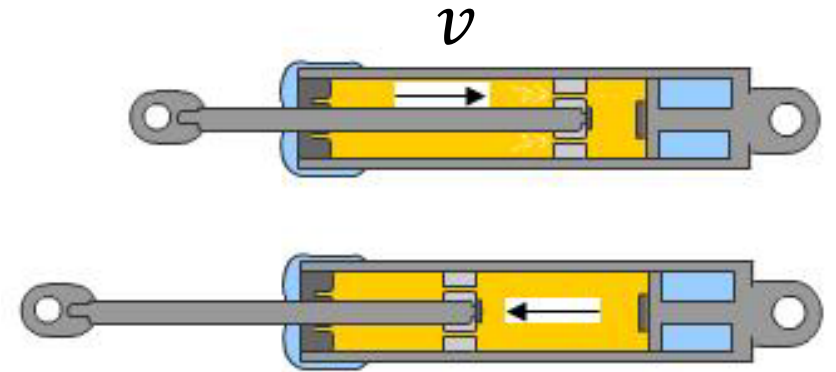
Spring and Damper

- Spring 
- Hooke's law



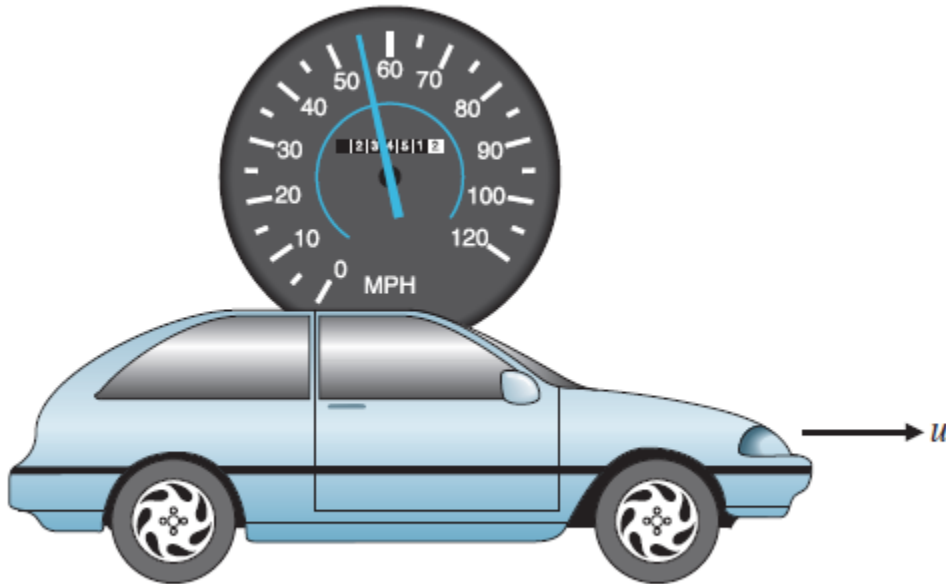
$$F_s = -kx$$

- Viscous Damper 
- Model friction



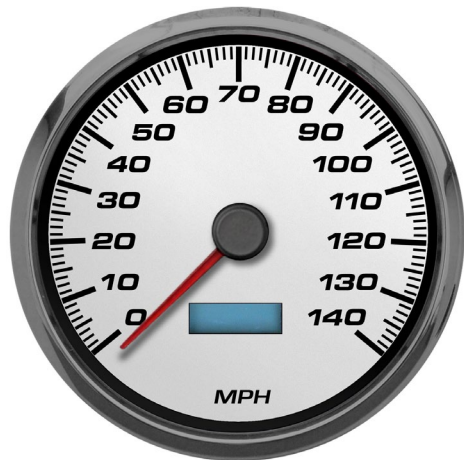
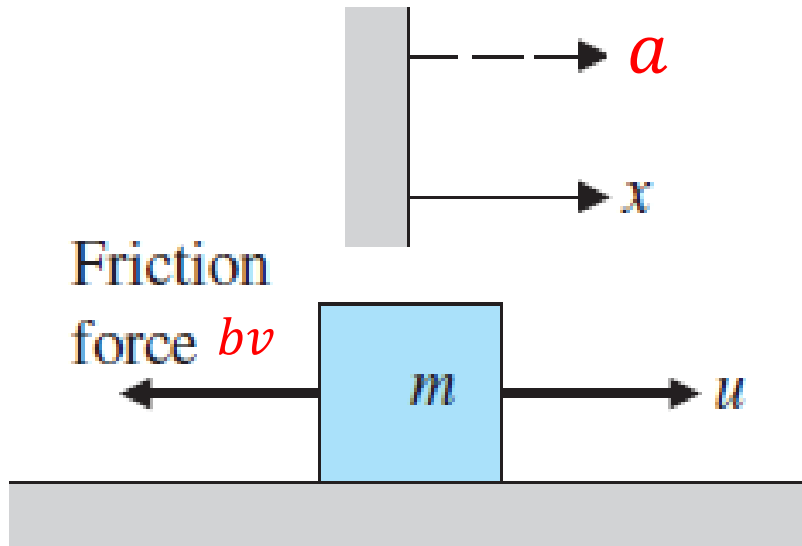
$$F_d = -bv$$

Cruise Control Model



- Write the **equation of motion** for the speed and forward motion
- Find the model between the **engine force** u (input) and the **speed** v (output)
- Input jumps from being $u=0$ at time $t=0$ to a constant $u=500$ N thereafter.
- Car mass m is 1000 kg and viscous drag coefficient $b= 50$ N*sec/m.

Free-body diagram for cruise control



- The result is

$$u(t) - bv(t) = ma(t)$$

- Input-Output equation of motion

$$mv^{(1)}(t) + bv(t) = u(t)$$

- Transfer function Model

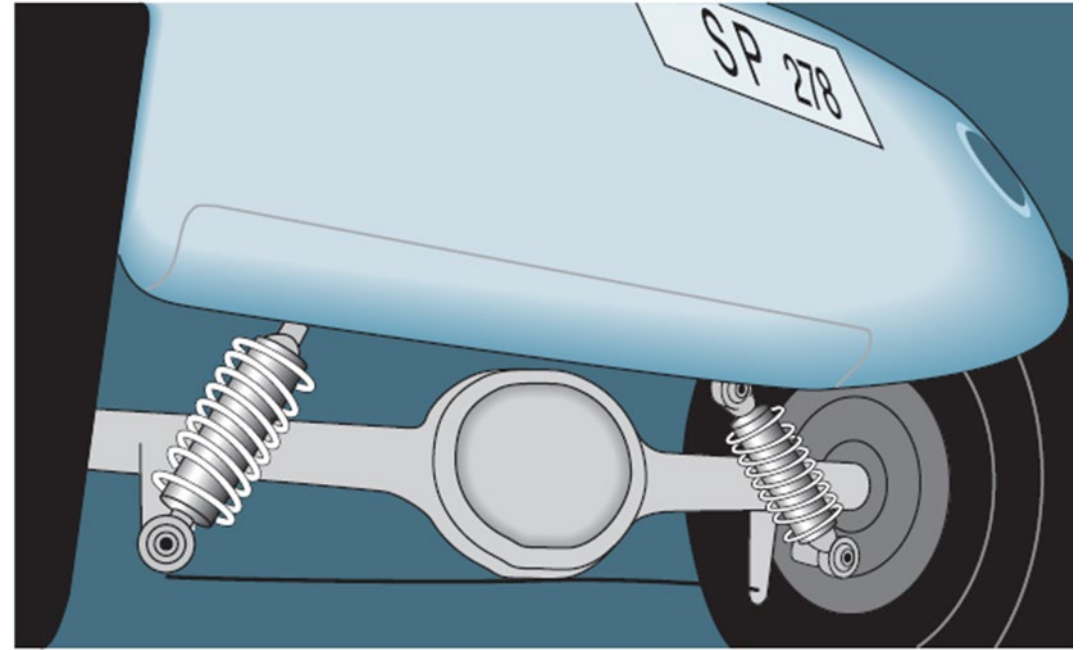
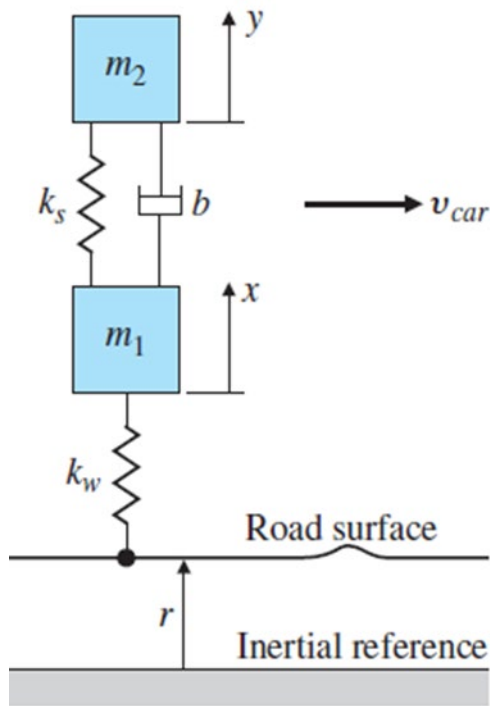
$$\frac{V(s)}{U(s)} = \frac{1}{ms + b}$$

Active Suspension System



Suspension Model

- Write the equation of motion for the automobile and wheel motion
- Vehicle mass: 1580 kg including four wheels (mass: 20 kg each)



- Car deflection: $k_s = 130,000$ N/m
- Wheel deflection: $k_w = 1,000,000$ N/m
- Friction constant: $b = 9800$ N*sec/m

Rotational Motion

- **Newton's law** to one-dimensional rotational systems

$$\tau = I\alpha$$

- **Torque τ** (N*m): normally applied from a motor
- **Moment of inertia I** (kg*m²)
- **Angular velocity ω** (rad/sec) and **angular displacement θ** (rad)

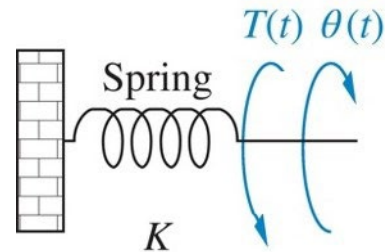
$$\omega(t) = \dot{\theta}^{(1)}(t)$$

$$\Omega(s) = s\Theta(s)$$

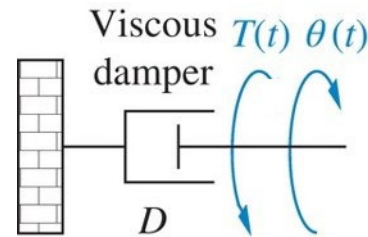
- **Angular acceleration α** (rad/sec²):

$$\alpha(t) = \dot{\omega}^{(1)}(t) = \ddot{\theta}^{(2)}(t)$$

$$A(s) = s\Omega(s) = s^2\Theta(s)$$



$$T(t) = -K\theta(t)$$



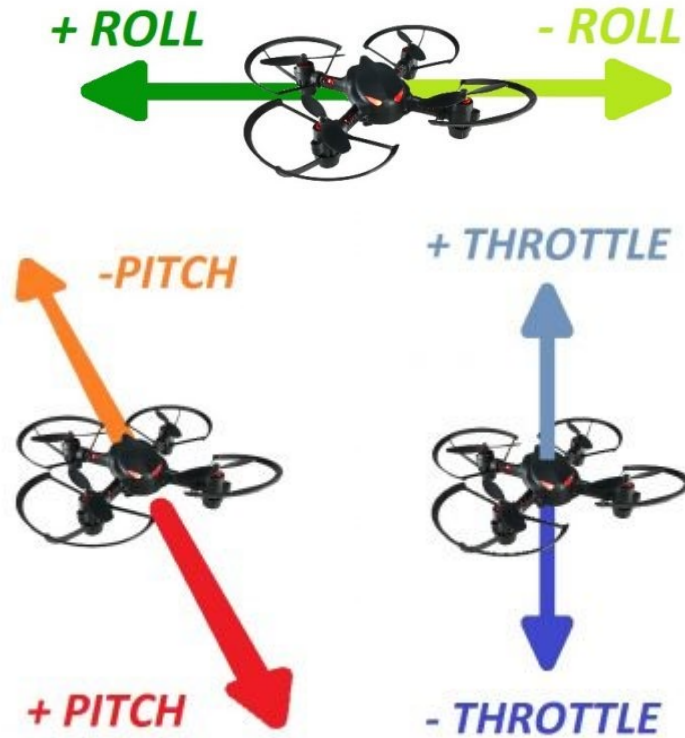
$$T(t) = -D\dot{\theta}^{(1)}(t)$$

Quadrotor Drone



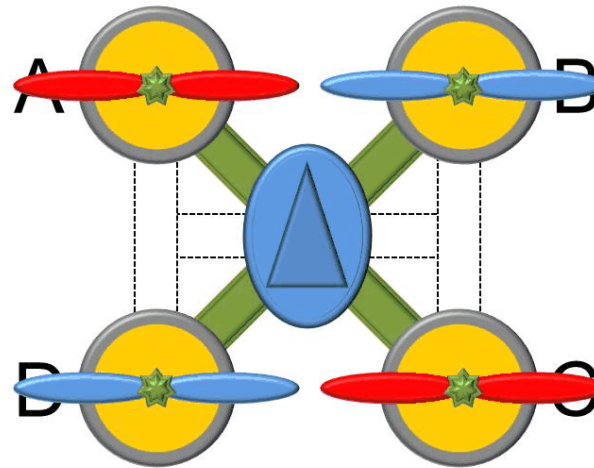
Pixhawk 6

Translational Motion



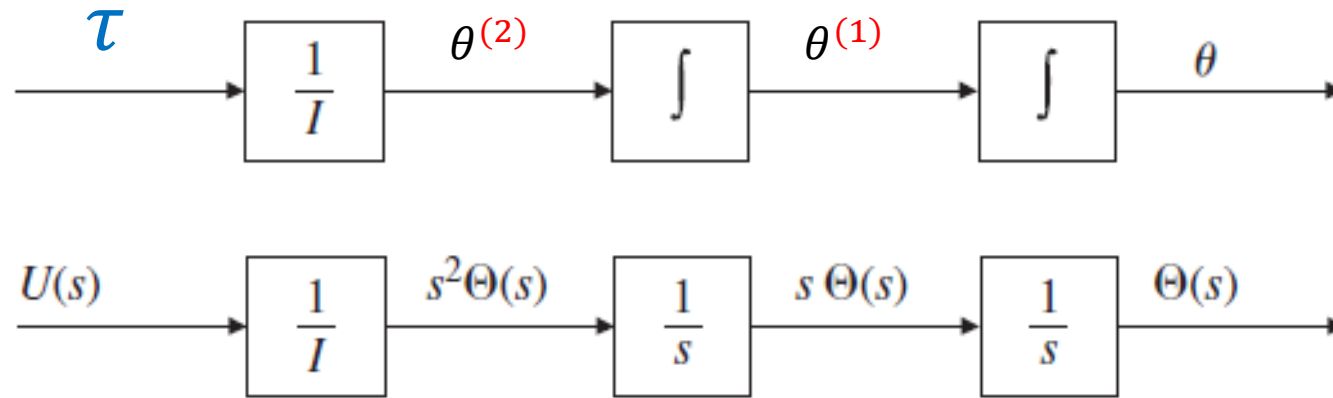
Rotational Motion

- **Roll**: Rotate around the front-to-back axis
- **Pitch**: Rotate around the side-to-side axis
- **Yaw**: Rotate around the vertical axis



Transfer function

- Transfer Function $\frac{\Theta(s)}{U(s)} = \frac{1}{I} \frac{1}{s^2}$ $\tau = I\theta^{(2)}$

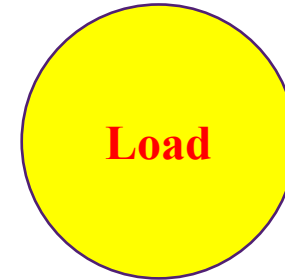
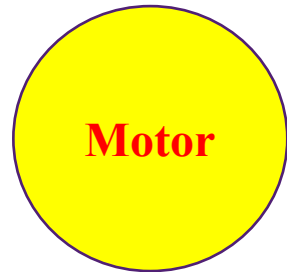
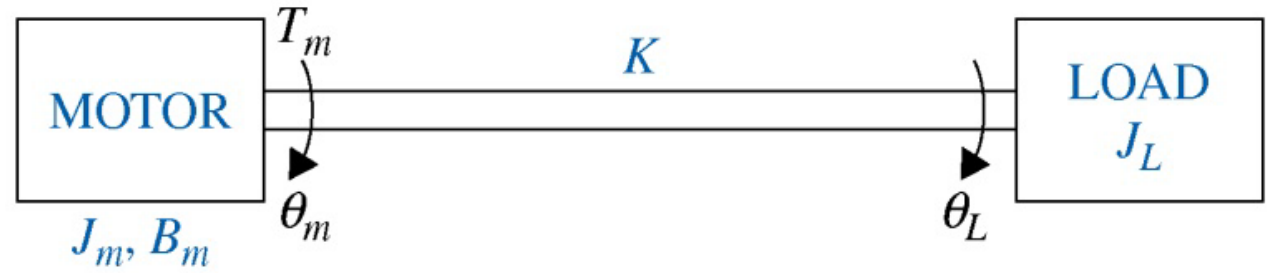


- Double-integrator plant

$$\frac{1}{s^2}$$

Motor-load system

- Flexible Connection



$$T_m(t) + K\theta_L(t) - B_m\theta_m^{(1)}(t) - K\theta_m(t) = J_m\theta_m^{(2)}(t)$$

$$K\theta_m(t) - K\theta_L(t) = J_L\theta_L^{(2)}(t)$$

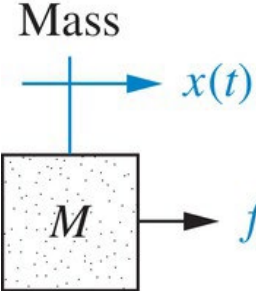
- Rigid Connection

$$T_m(t) - B_m\theta_m^{(1)}(t) = (J_m + J_L)\theta_m^{(2)}(t)$$

Summary

Translational Motion

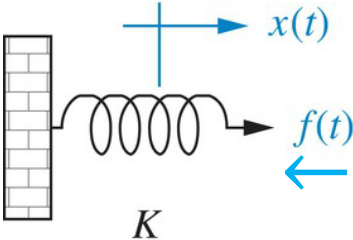
- Newton's law



Mass M

$$f(t) = Ma(t) = Mv^{(1)}(t) = Mx^{(2)}(t)$$

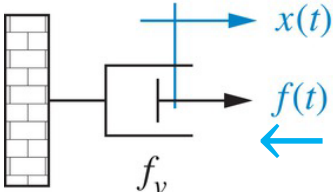
$$F(s) = MA(s) = MsV(s) = Ms^2X(s)$$



Spring K

$$f(t) = -Kx(t)$$

$$F(s) = -KX(s)$$



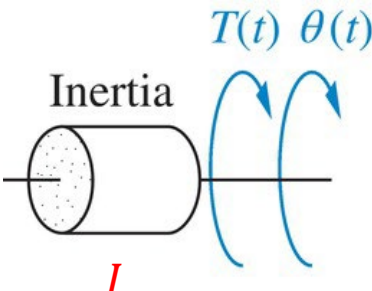
Viscous damper f_v

$$f(t) = -f_v x^{(1)}(t)$$

$$F(s) = -f_v sX(s)$$

Rotational Motion

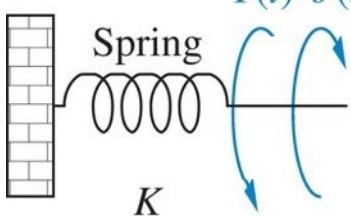
- Newton's law



Inertia I

$$T(t) = I\alpha(t) = I\omega^{(1)}(t) = I\theta^{(2)}(t)$$

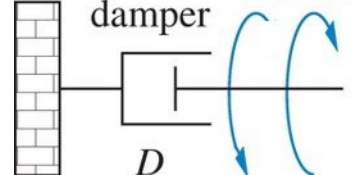
$$T(s) = IA(s) = Is\Omega(s) = Is^2\Theta(s)$$



Spring K

$$T(t) = -K\theta(t)$$

$$T(s) = -K\Theta(s)$$



Viscous damper D

$$T(t) = -D\theta^{(1)}(t)$$

$$T(s) = -Ds\Theta(s)$$