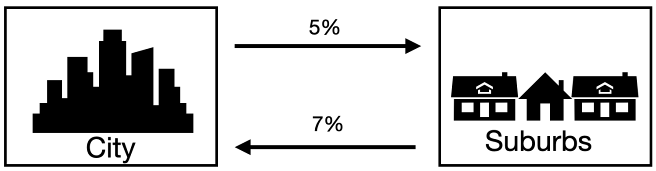
Background Preparation for Part 2

Included here are two motivating examples and two assigned homework problems related to dynamical systems that the author has used to introduce this project.

Eigenvalue Motivating Examples from Class

These are two of the motivating examples covered in class at the beginning of the eigenvalue unit.

Motivation 1: Population Movement



- Yearly population movement between a city and the surrounding suburbs
  - 5% of the city population moves to surrounding suburbs
  - 7% of the suburb population moves into city

p_k = population in city in year k, s_k = population in suburbs in year k

System of Linear (Difference) Equations

$$p_{k+1} = 0.95p_k + 0.07s_k$$
$$s_{k+1} = 0.05p_k + 0.93s_k$$

Matrix-Vector (Difference) Equation

$$\begin{bmatrix} p_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} 0.95 & 0.07 \\ 0.05 & 0.93 \end{bmatrix} \begin{bmatrix} p_k \\ s_k \end{bmatrix}$$

Question: Is there a starting population level such that the populations stay stable?

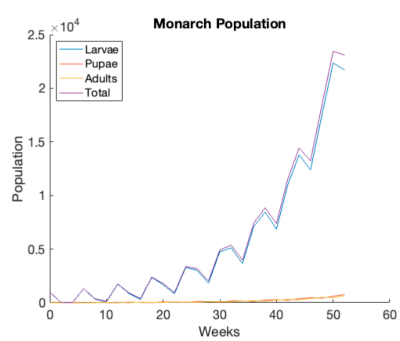
Motivation 2: Butterfly Population

System of Linear (Difference) Equations
(Population survival rates)

$$l_{k+1} = 45a_k$$
$$p_{k+1} = 0.03425l_k$$
$$a_{k+1} = 0.85l_k + 0.2458a_k$$

Matrix-Vector (Difference) Equation

$$\begin{bmatrix} l_{k+1} \\ j_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 45 \\ 0.03425 & 0 & 0 \\ 0 & 0.85 & 0.2458 \end{bmatrix} \begin{bmatrix} l_k \\ p_k \\ a_k \end{bmatrix}$$



Question: Can we use linear algebra to determine the long-term survival of the population?

Two homework problems on Dynamical Systems

These two homework problems are assigned in the week leading up to the project. The section numbers correspond to David Lay's Linear Algebra and its Applications, 6th edition.

1. (Section 5.1) Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ (lambda) and μ (mu). Let c_1 and c_2 be scalars. Define

$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}$$

for $k = 0, 1, 2, \ldots$

(a) Use the given definition above to find a formula for $\mathbf{x}_{k+1}$.

Solution. *We just need to replace k with $k+1$ in the given formula:*

$$\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}.$$

(b) Show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$.
(Hint: Think about how you can use eigenvalues and eigenvectors here!)

Solution.

$$\begin{aligned} A\mathbf{x}_{k+1} &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) && \text{(definition of } \mathbf{x}_k\text{)} \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} && \text{(distribute } A\text{)} \\ &= c_1\lambda^k(\lambda\mathbf{u}) + c_2\mu^k(\mu\mathbf{v}) && \text{(use eigenvector equations for } \mathbf{u} \text{ and } \mathbf{v}\text{)} \\ &= c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v} && \text{(simplify)} \end{aligned}$$

(c) Suppose that $|\lambda| < 1$ and $|\mu| < 1$. For this scenario, explain why $\mathbf{x}_k \to \mathbf{0}$ as $k \to \infty$, regardless of the choice of scalars c_1 and c_2.

Solution. *Since $|\lambda| < 1$ and $|\mu| < 1$, λ^k and μ^k will get smaller and approach 0 as $k \to \infty$. Thus $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v} \to c_1(0)\mathbf{u} + c_2(0)\mathbf{v} = \mathbf{0}$ as $k \to \infty$.*

2. Consider the following difference equation model representing the population movement between a city and the surrounding suburbs. Suppose that 5% of the city population each year moves to the surrounding suburbs, while 7% of the suburb population moves into the city each year. Assume these migration rates remain constant from year to year.

Let p_k represent the population in the city in year k and s_k the population in the suburbs in year k. Assume population is measured in millions, so that $p_k = 1$ means 1 million people. The following difference equation represents how the populations change year-to-year according to this model.

$$\begin{aligned} p_{k+1} &= 0.95p_k + 0.07s_k \\ s_{k+1} &= 0.05p_k + 0.93s_k \end{aligned}$$

Remember, you encountered a difference equation in Mini-Project 1. For a refresher, look back at the Mini-Project 1 reading on pp. 89-90 at the end of Section 1.10.

(a) In a few sentences, explain how to get from the given information to the difference equation.

Solution. *Each year, 95% of the city population stays in the city ($0.95p_k$), while 7% of the suburb population enters the city ($0.07s_k$). This gives us the p_{k+1} equation. For the second equation, we know that 5% of the city population enters the suburbs ($0.05p_k$) while 93% of the suburb population stays in the suburbs ($0.93s_k$).*

(b) Express the difference equation in matrix-vector form $\mathbf{x}_{k+1} = M\mathbf{x}_k$. We call the matrix in this model a *migration matrix*. Show that $\lambda_1 = 1$ is an eigenvalue of M and find its corresponding eigenvector, $\mathbf{v}_1$. What do the components of this vector represent in context of the model?

Solution. *We have* $M = \begin{bmatrix} 0.95 & 0.07 \\ 0.05 & 0.93 \end{bmatrix}$.

To show that 1 is an eigenvalue, you can solve $M\mathbf{v}_1 = \mathbf{v}_1$, *or* $(M - I)\mathbf{v}_1 = \mathbf{0}$.

$$[M - I|\mathbf{0}] = \left[\begin{array}{cc|c} -0.05 & 0.07 & 0 \\ 0.05 & -0.07 & 0 \end{array}\right] \xrightarrow[rref]{} \left[\begin{array}{cc|c} 1 & -7/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Solving this equation, we find $\mathbf{v}_1 = \begin{bmatrix} 7/5 \\ 1 \end{bmatrix}$ *(or any nonzero multiple of this). This tells us that if we start with* $7/5 = 1.4$ *million people in the city and 1 million people in the suburbs, the populations will remain unchanged from year to year. This is because* $\mathbf{x}_{k+1} = M\mathbf{x}_k$, *and so when* $\mathbf{x}_0 = \mathbf{v}_1$, $\mathbf{x}_1 = M\mathbf{v}_1 = \mathbf{v}_1 = \mathbf{x}_0$, *which means* $\mathbf{x}_2 = M\mathbf{x}_1 = M\mathbf{v}_1 = \mathbf{v}_1 = \mathbf{x}_0$, *and so on.*

(c) Find a basis for $\mathbb{R}^2$ consisting of $\mathbf{v}_1$ and another eigenvector, $\mathbf{v}_2$, of A.

Solution. *We need to find the other eigenvalue, which means we need to compute the roots of the characteristic polynomial.*

$$\begin{aligned} p(\lambda) = \det(M - \lambda I) &= \begin{vmatrix} 0.95 - \lambda & 0.07 \\ 0.05 & 0.93 - \lambda \end{vmatrix} \\ &= (0.95 - \lambda)(0.93 - \lambda) - 0.0035 \\ &= \lambda^2 - 1.88\lambda + 0.88 \end{aligned}$$

Thus $\lambda = \dfrac{1.88 \pm \sqrt{1.88^2 - 4(0.88)}}{2} = \dfrac{1.88 \pm 0.12}{2} = 1,\ 0.88$. *Solving* $(M - 0.88I)\mathbf{v}_2 = \mathbf{0}$, *we find* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *(or any nonzero multiple of this). These two eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *form a basis for* $\mathbb{R}^2$.

(d) Suppose that initially the total population is one million people and is split equally between the city and the surrounding suburb. Find c_1 and c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

Solution. *We need to solve*

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = c_1 \begin{bmatrix} 7/5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution to this system is $c_1 = 5/12$ *and* $c_2 = 1/12$.

(e) Given $\mathbf{x}_0$ from (d), find a formula for $\mathbf{x}_k$ (look back at the Feedback Problem 5 in Section 5.1). Then, determine what happens to the populations as $k \to \infty$ according to this model.

Solution. *We have* $\mathbf{x}_0 = 5/12\mathbf{v}_1 + 1/12\mathbf{v}_2$. *From problem 5 in Section 5.1 feedback problems, we have*

$$\mathbf{x}_k = (5/12)1^k\mathbf{v}_1 + (1/12)(0.88)^k\mathbf{v}_2.$$

As $k \to \infty$, $(0.88)^k \to 0$. *Also,* $1^k = 1$ *for all* k. *Thus* $\mathbf{x}_k \to (5/12)\mathbf{v}_1 = \begin{bmatrix} 7/12 \\ 5/12 \end{bmatrix}$. *The first value represents the limit of the population in the city as* $k \to \infty$ *and the second the population in the suburbs, assuming that we start with half a million in each region.*