Part 2: Analysis of the Monarch Population Matrix Model

In this activity, you will analyze the monarch population model that you previously studied in Part 1 earlier this semester. In particular, you will see how eigenvalues and eigenvectors are key to characterizing long-term outcomes.

Learning Goals

- Discover the relationships between eigenvectors/eigenvalues and long-term outcomes of the Monarch population model by using MATLAB to generate and examine model data.
- Explain these relationships using the language and theory of linear algebra.
- Exactly determine the critical survival rate for a given life-stage using linear algebra.
- Apply this analysis to explain the long-term outcomes of another species of choice using this model.

Background

Before you begin, carefully review the Monarch Population Model activity from earlier this semester. Also review classwork and homework you completed on dynamical systems. Below we summarize a brief introduction to dynamical systems to set the stage for this activity.

A (discrete) **dynamical system** is a mathematical model that describes how something changes step-by-step over time. We are interested in linear dynamical systems. These have the form $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where A is an $n \times n$ square matrix. The vectors $\mathbf{x}_k$ are called **state vectors** and the components are **state variables**. The matrix A is called the **transition** matrix.

Assume that our transition matrix A is diagonalizable. Arrange the eigenvalues in decreasing order: $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the corresponding basis of eigenvectors. For a given initial state vector, $\mathbf{x}_0$, let $c_1, c_2, \ldots, c_n$ be the coefficients so that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n.$$

Utilizing the distributive property and the eigenvector equation $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, we have

$$\begin{aligned}
\mathbf{x}_1 &= A\mathbf{x}_0 \\
&= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n) \\
&= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \ldots c_nA\mathbf{v}_n \\
&= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots c_n\lambda_n\mathbf{v}_n
\end{aligned}$$

Repeating this process, we find that $\mathbf{x}_2 = c_1\lambda_1^2\mathbf{v}_1 + c_2\lambda_2^2\mathbf{v}_2 + \ldots c_n\lambda_n^2\mathbf{v}_n$. Continuing this pattern, it follows that our difference equation can be rewritten in terms of the eigenvectors and eigenvalues of A:

$$\begin{aligned}
\mathbf{x}_k &= A\mathbf{x}_{k-1} \\
&= c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + \ldots c_n(\lambda_n)^k\mathbf{v}_n
\end{aligned}$$

This is called the **eigenvector decomposition** of our difference equation. In this activity, you will utilize this eigenvector decomposition to study the long-term behavior of the Monarch Population Model.

Activity

You will complete this activity in MATLAB. Open the template `monarch_population_analysis.mlx` and save it in your class folder. Each problem has it's own section in this template.

1. First, recreate the plot of the Monarch population (by life stage), starting with 1000 larvae, but this time use a 2.8% larval survival rate (let's pretend our sanctuary is currently dealing with some larva-eating ants). Generate and plot 2 years of data. You can do this by utilizing the code from Problems 5-6 in the previous activity and extending out the time.

2. Find the eigenvalues of A using MATLAB's `eig` command. Thinking back to what you know about eigenvalues and dynamical systems, what do you think this suggests about the long-term outcomes of the model? Does this agree with what you observed in Problem 1?

3. Let's look more closely at how the population is distributed amongst the three life-stages. To do this, consider the **distribution vector**, $\mathbf{d}_k = \mathbf{x}_k/t_k$, where $t_k = l_k + p_k + a_k$ (t_k is the total population at time level k). Copy and paste your code from Problem 1 into this cell. Incorporate a distribution vector, `d` into your `for`-loop and build a distribution matrix, `D` $= [\mathbf{d}_0 \quad \mathbf{d}_1 \quad \ldots \quad \mathbf{d}_{52}]$ similar to how you built the matrix `P`. Then create a plot of the population distribution amongst the life stages over the course of two years. What do you observe here?

4. Now let's examine the **ratios** of two consecutive populations in each life stage. That is, we want to look at l_{k+1}/l_k, p_{k+1}/p_k, and a_{k+1}/a_k for $k = 1, \ldots 51$. Thankfully, in MATLAB we can do this quickly by creating a ratio matrix, `Ratios = P(:,2:end)./P(:,1:end-1)`. Think about why this command will calculate all of our ratios at once and explain how it works. Why is the `.` important in the command?

5. Plot the ratios of each life stage. What do you observe? It may help to output the final ratios. Look back at the eigenvalues you calculated in problem 2. Do you see any connection?

6. Now go back and use MATLAB to calculate the eigenvector corresponding to the largest eigenvalue of A. Convert this eigenvector into a distribution eigenvector (divide each entry by the sum of the entries). Compare it to what you observed in Problem 3. Write a few sentences summarizing the relationships you are seeing between the eigenvalues, eigenvectors, distribution vectors, and ratios.

7. Now think about the critical larval survival rate question (Problem 8 in Part 1). Explain why this corresponds to when the largest eigenvalue of A is exactly equal to one.

8. Let's find this critical survival rate using linear algebra. Replace the larval survival rate in the matrix A with the variable L. Find the value of L such that $\det(A - I) = 0$. Call this value L_c. Explain why this is the value of L that results in $\lambda = 1$ being an eigenvalue. How does L_c compare to the experimental value you found in Part 1 Problem 8?

9. Examine the eigenvalues and eigenvectors when $L = L_c$ and use them to explain what will happen to the sanctuary's Monarch population in the long term. Provide a visualization of your results (you decide what visualization is most meaningful).

10. In Part 1 Problem 9 of the Monarch Population activity, you presented and explored another species that could be modeled by life-stages using population matrices. Now apply the eigenvalue/eigenvector analysis to that species' population model. Illustrate your conclusions with visualizations of your choice. What does your analysis tell you about the long-term outcome of the population?