Conduction and Convection in a Fin of Uniform Cross Section

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1 Problem Statement

Conduction and convection are two important modes of heat transfer. To understand the mechanisms of these two types of heat transfer processes, we study the temperature distribution in a fin of uniform cross section through analytical and numerical approaches. Various boundary conditions are investigated to demonstrate the effects of conduction and convection on the temperature distribution. Some of the related engineering applications of this study include the design of a heat exchanger with fins or a thermal radiator.

2 Learning Outcomes

Upon completing this project, students should learn to solve the temperature distribution along a one-dimensional fin analytically and numerically. Specifically, students learn to solve diffusion equations with source using the finite volume method and implement the algorithm via MATLAB programming. Students are expected to build a MATLAB App to perform parametric studies and to visualize the simulation results.

3 Context for Use

This activity can be used in junior/senior level engineering courses such as heat transfer and computational fluid dynamics or in a physics course on the concepts of conduction and convection. It requires students to investigate the temperature distribution along a one-dimensional fin through analytical and numerical approaches. Several parameters and boundary conditions are to be investigated, which makes the study comprehensive and challenging. Students are expected to complete the project within one week including modeling and simulation. A written report is required to address all questions and exercises in the activity.

The prerequisites of this project are some familiarity with undergraduate-level heat transfer, ordinary differential equations, and numerical analysis. Students are expected to know some common techniques in solving ordinary differential equations analytically. Also, students need to understand the basic procedures in solving diffusion equations with source, numerically using the finite difference or finite volume method. This activity provides students with the opportunity to develop their computational skills on MATLAB programming and MATLAB App building.

4 Mathematical Model

The heat transfer process in a fin of uniform cross section is governed by two physical laws: Fourier's law and Newton's law of cooling [1]. Assume the fin has a length of L and a constant cross-sectional area A. The temperature at the base ($x = 0$) of the fin is fixed at $T(0) = T_b$. The right end ($x = L$) of the fin is subject to varying boundary conditions. The lateral surface of the fin is exposed to a moving fluid at temperature T_{fluid} with a convective heat transfer coefficient h. If we take a finite element of length dx from the fin, this small slice of material undergoes conduction along the axial direction and convective cooling/heating on the external surface. Assume the conductive heat transfer rates on the two sides of the element are q_x and q_{x+dx} and the differential convection rate through the side surface is dq_s (see Figure 1).

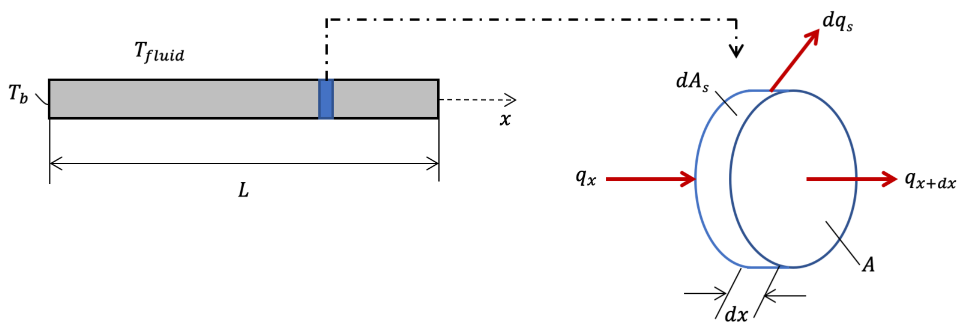

Figure 1. Schematic of energy balance for an element of the fin

Applying the principle of conservation of energy to the finite element gives:

$$q_x = q_{x+dx} + dq_s. \tag{1}$$

From Fourier's law [1], we know that $q_x = -kA\frac{dT}{dx}$ where k is the thermal conductivity of the material. The heat rate at the right side of the element can be approximated using a 1st-order Taylor series expansion $q_{x+dx} = q_x + \frac{dq_x}{dx}dx$. The differential heat transfer rate dq_s due to convection is determined by the Newton's law of cooling [1] $dq_s = hdA_s(T - T_{fluid})$ where dA_s is the lateral surface area of the finite element.

Substituting the above quantities into (1), we can derive the governing equation for temperature distribution $T(x)$ in a fin as follows:

$$\frac{d^2T}{dx^2} - \frac{hp}{kA}(T - T_{fluid}) = 0, \tag{2}$$

where p is the perimeter of the cross section and $p = \frac{dA_s}{dx}$. The temperature distribution $T(x)$ is affected by the thermal conductivity k of the material, the perimeter p and area A of the cross section, the ambient fluid temperature T_{fluid}, and the convective heat transfer coefficient h.

If we define a new temperature variable $\Phi(x) \equiv T(x) - T_{fluid}$, (2) reduces to

$$\frac{d^2\Phi}{dx^2} - C^2\Phi = 0, \tag{3}$$

where $C^2 = \frac{hp}{kA}$. It can be verified that the solution of (3) takes the form of $\Phi(x) = Ae^{Cx} + Be^{-Cx}$. Here the coefficients A and B can be evaluated using the boundary conditions at the two ends of the fin.

Now we need to specify the boundary conditions. According to the experiments, the temperature at the base ($x = 0$) is fixed by $T(0) = T_b$. Thus, $\Phi(0) \equiv T(0) - T_{fluid} = T_b - T_{fluid} \equiv \Phi_b$. In accordance with the experimental setups, the boundary condition at the tip ($x = L$) can be specified by one of the following:

- A prescribed temperature $\Phi(L) = \Phi_L$.
- A prescribed flux $\frac{d\Phi}{dx}_{|x=L} = q_f$. If $q_f = 0$, it is an insulated (adiabatic) tip.
- A convective condition $-k\frac{d\Phi}{dx}_{|x=L} = h\Phi(L)$.

Then, the temperature distribution $\Phi(x)$ can be obtained from solving (3) with boundary conditions.

4.1 Example

Consider a slender, finite-length fin with a convective tip. Derive the expression of the temperature distribution in the fin.

From the above analysis, we know that the solution can be represented by $\Phi(x) = Ae^{Cx} + Be^{-Cx}$ with unknown coefficients A and B. The two boundary conditions are $\Phi(0) = \Phi_b$ and $-k\frac{d\Phi}{dx}_{|x=L} = h\Phi(L)$. Substituting $\Phi(x)$ into boundary conditions gives:

$$\begin{aligned} \Phi_b &= A + B \\ -kC(Ae^{CL} - Be^{-CL}) &= h(Ae^{CL} + Be^{-CL}). \end{aligned} \tag{4}$$

The coefficients A and B can be determined by solving the equation system (4). Then, the temperature distribution $\Phi(x)$ in the fin is

$$\Phi(x) = \frac{\cosh C(L-x) + \frac{h}{Ck}\sinh C(L-x)}{\cosh CL + \frac{h}{Ck}\sinh CL}\Phi_b. \tag{5}$$

A plot of the temperature $T(x) = T_{fluid} + \Phi(x)$ in a fin with specified geometric and thermophysical conditions can be found by the curve of the analytical solution in §5.

4.2 Exercises

(i) Solve for the temperature distribution in a fin when the tip temperature is prescribed at $T(L) = T_{tip}$.

(ii) Solve for the temperature distribution in a fin when the tip is insulated, i.e., $\frac{d\Phi}{dx}_{|x=L} = 0$.

(iii) Replace the above circular fin by a rectangular fin with the same cross-sectional area. Discuss how this shape change affects the temperature distribution.

5 Numerical Simulations

The steady-state equation (2) with boundary conditions is solved numerically using the finite volume method [2]. The computational domain is divided into N node-centered control volumes or cells, as shown in Figure 2. The grid has a uniform mesh size of $\Delta x = \frac{L}{N}$. Integration of (2) over a control volume gives

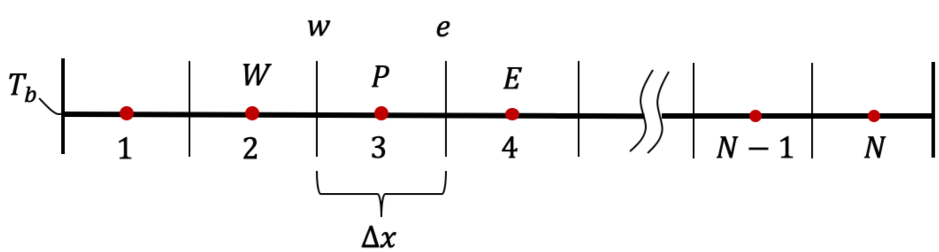

Figure 2. The grid

$$\int_S \frac{dT}{dx}\vec{n}\,dS - \int_{\mathcal{V}} C^2(T - T_{fluid})d\mathcal{V} = 0, \tag{6}$$

where the integrals in (6) are evaluated locally within each control volume. The first surface integral represents the heat flux through the surfaces of a control volume. The second volume integral can be regarded as a source term within the control volume. For example, evaluating (6) over control volume 3 gives

$$\left(\frac{dT}{dx}\right)_e - \left(\frac{dT}{dx}\right)_w - C^2(T_P - T_{fluid})\Delta x = 0. \tag{7}$$

Here the subscripts e and w refer to the east and west surfaces of the control volume. P represents the center of the control volume. The two temperature gradients in (7) can be evaluated using linear approximations with temperatures at neighboring cells 2 and 4. Therefore, for internal cells from 2 to $N - 1$, replacing the temperature gradients in (7) gives

$$\frac{T_E - T_P}{\Delta x} - \frac{T_P - T_W}{\Delta x} - C^2(T_P - T_{fluid})\Delta x = 0, \tag{8}$$

where the subscripts E and W refer to the center nodes of the east and west cells around the P cell. For boundary cells 1 and N, the temperature gradients in (7) need to be evaluated using the given boundary conditions (see Appendix A in §6). Then, we can establish a system of N discretized equations. The temperature distribution $T(x)$ at each cell node can be calculated by solving the system of linear equations.

Figure 3 shows an interactive user interface for simulating the temperature distribution in a fin with varying tip conditions. On the left panel, the dimensions (length L, cross-sectional area A and perimeter p) of the fin and the thermophysical properties (thermal conductivity k and convective heat transfer coefficient h) can be entered. The ambient temperature condition is determined by the fluid temperature (Fluid T). For boundary conditions, the temperature on the left is prescribed at a base temperature (Base T). The condition on the right can be set up from the dropdown list and the corresponding tip temperature (Tip T) or flux (Tip Flux) is to be entered. On the right panel, users can input data from experiments. The middle panel shows the result of temperature distribution in the fin. The top plot shows the comparison of results from simulation (blue line with square markers), analytical solution (red line), and experiment (yellow circles). The error indicates the difference between the numerical solution and the exact solution. The bottom scales present the temperature distribution from the simulation.

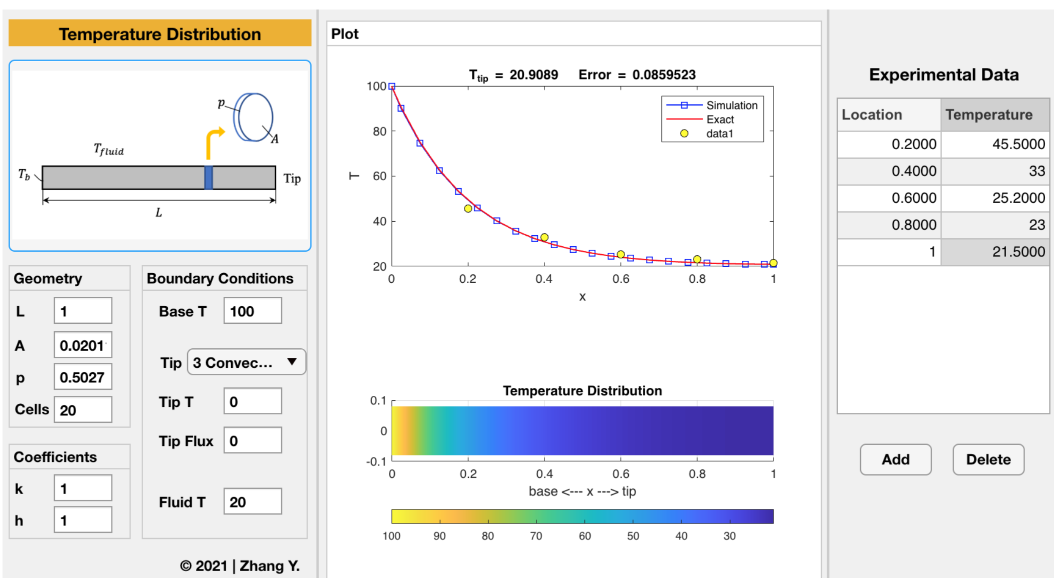

Figure 3. Temperature distribution MATLAB App

5.1 Numerical Exercises

(i) Investigate the interaction between conduction and convection by changing the values of thermal conductivity k and convective heat transfer coefficient h. If $k \gg h$, what is the temperature distribution when the tip is maintained at a constant room temperature? Explain the corresponding physical condition for this case. What happens if $k \ll h$? How is heat transferred in the fin?

(ii) Assume that the ambient fluid temperature remains constant. If a circular fin is under the convective tip condition, how could one increase the heat transfer rate by changing other parameters? Then, replace the above circular fin by a square bar with the same cross-sectional area. Discuss how this shape change affects the temperature distribution, especially the tip temperature.

(iii) What is the difference between the adiabatic and the convective tip conditions? To reach the same tip temperature, how are these two conditions related to the length of the fin?

(iv) How could one improve the accuracy of a numerical solution, especially when sharp gradients exist in the solution?

6 Appendix A

Here are the discretized equations for boundary cells in the finite volume scheme. For cell 1, the temperature is fixed at T_b on the left:

$$\frac{T_E - T_P}{\Delta x} - \frac{T_P - T_b}{\Delta x/2} - C^2(T_P - T_{fluid})\Delta x = 0. \tag{A.1}$$

For cell N, the discretized equation depends on the boundary condition on the tip: (i) a prescribed temperature T_{tip},

$$\frac{T_{tip} - T_P}{\Delta x/2} - \frac{T_P - T_W}{\Delta x} - C^2(T_P - T_{fluid})\Delta x = 0; \tag{A.2}$$

(ii) a prescribed flux q_f,

$$\frac{q_f}{k} + \frac{T_P - T_W}{\Delta x} + C^2(T_P - T_{fluid})\Delta x = 0; \tag{A.3}$$

(iii) a convective condition $-k\frac{T_{tip}-T_P}{\Delta x/2} = h(T_{tip} - T_{fluid})$,

$$\frac{2h}{h\Delta x + 2k}(T_{fluid} - T_P) - \frac{T_P - T_W}{\Delta x} - C^2(T_P - T_{fluid})\Delta x = 0. \tag{A.4}$$

7 Appendix B: Hints for Selected Exercises

§4.2(i) We know that the solution can be represented by $\Phi(x) = Ae^{Cx} + Be^{-Cx}$ with unknown coefficients A and B. The two boundary conditions are $\Phi(0) = \Phi_b$ and $\Phi(L) \equiv T(L) - T_{fluid} = T_{tip} - T_{fluid} \equiv \Phi_L$. Substituting $\Phi(x)$ into boundary conditions gives:

$$\begin{aligned} \Phi_b &= A + B \\ \Phi_L &= Ae^{CL} + Be^{-CL}. \end{aligned} \tag{B.1}$$

The coefficients A and B can be determined by solving the equation system (B.1). Then, the temperature distribution in the fin is

$$\Phi(x) = \frac{\sinh C(L-x) + \frac{\Phi_L}{\Phi_b}\sinh Cx}{\sinh CL}\Phi_b. \tag{B.2}$$

§4.2(ii) This problem can be solved in a similar fashion. The two boundary conditions are $\Phi(0) = \Phi_b$ and $\frac{d\Phi}{dx}_{|x=L} = 0$. Substitute the general expression of $\Phi(x)$ into boundary conditions:

$$\begin{aligned} \Phi_b &= A + B \\ 0 &= C(Ae^{CL} - Be^{-CL}). \end{aligned} \tag{B.3}$$

The coefficients A and B can be determined by solving the equation system (B.3). Then, the temperature distribution in the fin is

$$\Phi(x) = \frac{\cosh C(L-x)}{\cosh CL}\Phi_b. \tag{B.4}$$

§5.1(i) For a fin with fixed temperatures on both ends, the effect of conduction dominates if $k \gg h$. The convective heat transfer between the lateral surface of the bar and the ambient fluid flow can be ignored. The temperature distribution has a linear trend. If $k \ll h$, it indicates the material has a very low thermal conductivity and there is a strong convection between the lateral surface of the bar and the ambient fluid flow.

§5.1(ii) The heat transfer rate can be increased by replacing the ambient fluid by fluids with higher convection heat transfer coefficients. Changing the circular rod to a square bar with the same cross-sectional area will slightly lower the temperature at the tip. This can be verified from the MATLAB App.

§5.1(iii) The convective tip condition assumes an energy balance between conduction and convection at the tip, while the adiabatic tip condition assumes convective heat transfer at the tip is negligible. Therefore, a bar with the adiabatic tip needs to be longer than a bar with the convective tip to reach the same tip temperature.

§5.1(iv) The accuracy of the numerical solution for this one-dimensional problem can be improved by using a finer uniform mesh or a non-uniform mesh where the grid is finer when sharp gradients exist.

REFERENCES

[1] Bergman, T. L., Incropera, F. P., DeWitt, D. P., & Lavine, A. S. 2011. *Fundamentals of heat and mass transfer*. John Wiley & Sons.

[2] Ferziger, J. H., Perić, M., & Street, R. L. 2002. *Computational methods for fluid dynamics.* Springer.