## Step Response of Parallel RLC Circuit

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- Topic: RLC Circuits


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## Introduction

The second-order differential equations with constant coefficients with constant input
$\frac{d^{2} v(t)}{\mathrm{dt}^{2}}+a_{1} \frac{\mathrm{dv}(t)}{\mathrm{dt}}+a_{0} v(t)=b_{0}$
The solution to Equation (1) is the sum of the complementary solution $\mathrm{vc}(\mathrm{t})$ and the particular solution $\mathrm{vp}(\mathrm{t})$

The complementary solution is the solution to homogeneous differential equation given
by
$\frac{d^{2} v(t)}{\mathrm{dt}^{2}}+a_{1} \frac{\mathrm{dv}(t)}{\mathrm{dt}}+a_{0} v(t)=0$
Depending on the coefficients, there are three cases -overdamped, critically damped, and underdamped as discussed earlier.

The coefficients of the complementary solution are found by applying the initial conditions to sum $\mathrm{vc}(\mathrm{t})+\mathrm{vp}(\mathrm{t})$ after finding the particular solution

The particular solution is the solution to the original differential equation given by Equation (1), including the input.

The form of the particular solution is similar to the input signal. For the constant input, the particular solution will be a constant. Let the particular solution be

$$
\begin{equation*}
v_{p}(t)=K \tag{4}
\end{equation*}
$$

Substituting this proposed solution to Equation (1), we obtain
$\frac{d^{2} K}{d t^{2}}+a_{1} \frac{d K}{d t}+a_{0} K=b_{0}$
Since K is a constant, $\mathrm{dK} / \mathrm{dt}=0$ and $\frac{d^{2} K}{\mathrm{dt}^{2}}=0$, and Equation (3) becomes $a_{0} K=b_{0}$
Thus, for constant input signal, the particular solution to Equation (1) is given by
$v_{p}(t)=K=\frac{b_{0}}{a_{0}}$

## Step response of Parallel RLC Circuit

A series RLC circuit with constant independent source is given in the following figure


Let the initial voltage across the capacitor at $\mathrm{t}=0$ be $V_{0}$, and the initial current through the inductor at $\mathrm{t}=0$ be $I_{0}$ Sum the current away from node a:
$-i_{s}+\frac{v(t)}{R}+i(t)+C \frac{d v(t)}{d t}=0$
The voltage across inductor
$v(t)=L \frac{d i(t)}{d t}$
Substitute (8) into (7)
$\left.-I_{s}+\frac{L}{R} \frac{d i(t)}{d t}+i(t)+L C \frac{d^{2} i(t)}{d t^{2}}=0\right)$
Rearrange (9)
$\frac{d^{2} i(t)}{d t^{2}}+\frac{1}{R C} \frac{d i(t)}{d t}+\frac{1}{L C} i(t)=\frac{1}{L C} I_{s}$
$a_{2}=1, a_{1}=\frac{1}{\mathrm{RC}}, a_{0}=\frac{1}{\mathrm{LC}}$
$\alpha=\frac{a_{1}}{2} \quad, \omega_{0}=\sqrt{\left(a_{0}\right)}=\frac{1}{\sqrt{\mathrm{LC}}}$
$s_{1}=-\alpha+\sqrt{\alpha^{2}-\omega_{0}^{2}} \quad$ and $s_{1}=-\alpha-\sqrt{\alpha^{2}-\omega_{0}^{2}}$
Particular solution
$i_{p}(t)=\frac{b_{0}}{a_{0}}=I_{s}$
Adding $\mathrm{vc}(\mathrm{t})+\mathrm{vp}(\mathrm{t})$. Then applying the initial conditions to find the coeffeffien s in comaplemenary solution. Initial Values:

$$
\begin{align*}
& i(0)=I_{0}  \tag{12}\\
& \frac{d i(0)}{d t}=\frac{v(0)}{L}=\frac{V_{0}}{L} \tag{13}
\end{align*}
$$

\% Select the element values and initial values
\%R=900\% Ohms
\%C=0.5 \% uF
\%L=100 \% mH
\%V0=10 \% V
\%I0=50 \% mA
\%Vs=0;

```
%L=L*10^-3; % Conversion to H
%C=C*10^-6; % Conversion to F
%I0=I0*10^-3; % Conversion to A
%components, Source, and Initial Values
C=1.6e-9;
L=40e-3;
R=2.5e3;
V0=5;
I0=2e-3;
Is=8e-3;
a1 = 1 / (R*C)
a1 = 250000
a0 = 1 / (L*C)
a0 = 1.5625e+10
```

```
b0=Is/(L*C)
b0 = 125000000
alpha = a1 / 2
alpha = 125000
w= sqrt(a0)
w = 125000
% Find the roots
s1 = -alpha + sqrt(alpha^2 - w^2)
s1 = -125000
s2 = -alpha - sqrt(alpha^2 - w^2)
s2 = -125000
```


## Case 1: if $\alpha>\omega_{0}$. Overdamped

the two roots of the characteristic equation are real and distinct.
Thus, the complementary solution $\mathrm{vc}(\mathrm{t})$ can be written as
$i(t)=A_{1} e^{s_{1} t}+A_{2} e^{s_{2} t}, t \geq 0$
The particular solution is given
$i_{p}(t)=\frac{b_{0}}{a_{0}}=I_{s}$
The total solution is written as
$i(t)=i_{p}(t)+i(t)=I_{s}+A_{1} e^{s_{1} t}+A_{2} e^{s_{2} t}, t \geq 0$
The coefficients are found by applying the initial conditions.
Setting $t=0$ in Equation (1), we obtain
$i(0)=I_{s}+A_{1}+A_{2} \quad->A_{1}+A_{2}=I_{0}-I_{s}$
Taking the derivative of the equation (1), we get
$\frac{d i(t)}{d t}=A_{1} s_{1} e^{s_{1} t}+A_{2} s_{2} e^{s_{2} t}$ and at $\mathrm{t}=0 \quad \frac{d v(0)}{d t}=A_{1} s_{1}+A_{2} s_{2}$
and $\frac{d i(0)}{d t}=\frac{v(0)}{L}=\frac{V_{0}}{L}$

Apply Equation (20) into (19)
as
Then, write in matrix form
$\left[\begin{array}{cc}1 & 1 \\ s_{1} & s_{2}\end{array}\right]\left[\begin{array}{l}A_{1} \\ A_{2}\end{array}\right]=\left[\begin{array}{c}I_{0}-I_{s} \\ \frac{V_{0}}{L}\end{array}\right]$, in matrxix form $\quad \mathrm{SA}=B, \quad A=S^{-1} B$

Finally $A_{1}=A(1), A_{2}=A(2)$

Case 2: if $\alpha=\omega_{0}$ Critically Damped
If $\alpha=\omega_{0}$, the two roots of the characteristic equation are real and equal.
$s_{1}=s_{2}=s=-\alpha=-\frac{a_{1}}{2}$
Since the solutions are identical, Thus, the complementary solution vc(t) can be written as
$i(t)=A_{1} e^{s t}+A_{2} t e^{s t}, t \geq 0$
The particular solution is given
$i_{p}(t)=\frac{b_{0}}{a_{0}}=I_{s}$
The total solution is written as
$i(t)=i_{p}(t)+i(t)=I_{s}+A_{1} e^{s t}+A_{2} t e^{s t}, t \geq 0$
The coefficients are found by applying the initial conditions.
Setting $t=0$ in Equation (5)
$i(0)=I_{s}+A_{1}=I_{0} \quad, \quad A_{1}=I_{0}-I_{s}$
Taking the derivative of Equation (4) and setting $t=0$
$\frac{d i(0)}{d t}=A_{1} s+A_{2}=\frac{V_{0}}{L}$
We obtain
$A_{2}=\frac{V_{0}}{L}-A_{1} s$
Case 3: if $\alpha<\omega_{0}$ Underdamped
The two roots are
$s_{1}=-\alpha+j \beta, \quad s_{2}=-\alpha-j \beta$
Thus, the complementary solution $i(t)$ can be written as
$i(t)=e^{-\alpha t}\left[B_{1} \cos (\beta t)+B_{2} \sin (\beta t)\right], t \geq 0$
The particular solution is given
$i_{p}(t)=\frac{b_{0}}{a_{0}}=I_{s}$
The total solution is written as
$i(t)=i_{p}(t)+i(t)=I_{s}+e^{-\alpha t}\left[B_{1} \cos (\beta t)+B_{2} \sin (\beta t)\right], t \geq 0$
The coefficients are found by applying the initial conditions.
Setting $t=0$ in Equation (8)
$i(0)=I_{s}+B_{1}=V_{0}, \quad B_{1}=I_{0}-I_{s}$
Taking the derivative of Equation (8) and setting $t=0$
$\frac{d i(0)}{d t}=-\alpha B_{1}+\beta B_{2}$ where $\frac{d i(0)}{d t}=\frac{V_{0}}{L}$
Substituting $i(0)=B_{1}$
$B_{2}=\frac{\frac{V_{0}}{L}+\alpha B_{1}}{\beta}$
if alpha^2 > w^2 \% overdamped
fprintf('case: overdamped $a^{\wedge} 2$ > wo^2');
\%Using initial values to find A1 and A2 coeffcients
$\mathrm{S}=[1 \mathrm{1}$;s1 s2]
b=[I0-Is;V0/L]
$A=S \backslash b$;
$\mathrm{A} 1=\mathrm{A}(1)$
$A 2=A(2)$ fprintf('i(t)= \%6.2f +\%6.2f $\exp (\% 6.2 f t)+\% 6.2 f \exp (\% 6.1 f t) m A t>=\theta^{\prime}, I s^{*} 1000, A 1^{*} 1000, s 1, A 2^{*}$
elseif alpha^2 < w^2
fprintf('case: underdamped $a^{\wedge} 2<$ wo $\left.^{\wedge} 2^{\prime}\right)$;
\%Using initial values to find A1 and A2 coeffcients beta=imag(s1)

```
    B1=I0-Is
    B2=(V0/L+alpha*B1)/beta
    i = Is + exp(-alpha*t).*(B1*cos(beta*t) + B2*sin(beta*t));
    fprintf('i(t)= %6.2f + exp(%6.2f t)[%6.2f cos(%6.1f t)+%6.2f sin(%6.2f t) mA, t>=0]',Is*10t
else
    fprintf('case: Critically damped alpha = w');
    %Using initial values to find A1 and A2 coeffcients
    s=-alpha
    A1=I0-Is
    A2=V0/L-A1*S
    %i = Is + A1*exp(s*t) + A2*t.*exp(s*t);
    fprintf('i(t)= %6.2f +%6.2f exp(%6.2f t)+%6.2f t exp(%6.1f t) mA t>=0',Is*1000,A1*1000,s,A
end
```

```
case: Critically damped alpha = w
```

case: Critically damped alpha = w
s = -125000
s = -125000
A1 = -0.0060
A1 = -0.0060
A2 = -625
A2 = -625
i(t)= 8.00 + -6.00 exp(-125000.00 t)+-625000.00 t exp(-125000.0 t) mA t>=0

```
i(t)= 8.00 + -6.00 exp(-125000.00 t)+-625000.00 t exp(-125000.0 t) mA t>=0
```


## Current $\mathrm{i}(\mathrm{t})$ and Voltages accross Resistor and Inductor .

Case 1: if $\alpha>\omega_{0}$. Overdamped
The voltage across the capacitor is given by
$v(t)=L \frac{d i(t)}{d t}=L\left(s_{1} A_{1} e^{s_{1} t}+s_{2} A_{2} e^{s_{2} t}\right), t \geq 0$
$v(t)=\left(C_{1} e^{s_{1}^{t}}+C_{2} e^{s_{2} t}\right), t \geq 0$, where $C_{1}=L s_{1} A_{1}$ and $C_{2}=L s_{2} A_{2}$
The current through the resistor
$i_{R}(t)=\frac{v(t)}{R}, t>0$
The current through the capacitor
$i_{C}(t)=C \frac{d \nu(t)}{d t}=C\left(s_{1} C_{1} e^{s_{1} t}+s_{2} C_{2} e^{s_{2} t}\right), t>0$
Case 2: if $\alpha=\omega_{0}$ Critically Damped
The current through the capacitor is given by
$v(t)=L \frac{d i(t)}{d t}=L\left(s A_{1} e^{s t}+A_{2} e^{s t}+s A_{2} t e^{s t}\right), t \geq 0$
$\left.v(t)=L\left(s A_{1}+A_{2}\right) e^{s t}+\operatorname{CsA}_{2} t e^{s t}\right), t \geq 0$,
$v(t)=\left(C_{1} e^{s t}+C_{2} t e^{s t}\right), t \geq 0$ where $C_{1}=L\left(s A_{1}+A_{2}\right)$ and $C_{2}=L\left(s A_{2}\right)$

The current through the resistor
$i_{R}(t)=\frac{v(t)}{R}, t>0$
The current through the capacitor
$i_{C}(t)=L \frac{d v(t)}{d t}=L\left(s C_{1} e^{s t}+C_{2} e^{s t}+s C_{2} t e^{s t}\right), t>0$
Case 3: if $\alpha<\omega_{0}$ Underdamped
The voltage across the capacitor is given by

$$
\begin{aligned}
& v(t)=L \frac{d i(t)}{d t}=L\left[-\alpha e^{-\alpha t}\left[B_{1} \cos (\beta t)+B_{2} \sin (\beta t)\right]+e^{-\alpha t}\left[-\beta B_{1} \sin (\beta t)+\beta B_{2} \cos (\beta t)\right]\right], t \geq 0 \\
& \left.v(t)=L\left[-\alpha B_{1}+\beta B_{2}\right] e^{-\alpha t} \cos (\beta t)-L\left[\alpha B_{2}+\beta B_{1}\right] e^{-\alpha t} \sin (\beta t)\right], t \geq 0 \\
& \left.v(t)=C_{1} e^{-\alpha t} \cos (\beta t)-C_{2} e^{-\alpha t} \sin (\beta t)\right], t \geq 0,
\end{aligned}
$$

where $C_{1}=L\left[-\alpha B_{1}+\beta B_{2}\right]$ and $C_{2}=L\left[\alpha B_{2}+\beta B_{1}\right]$
The current through the resistor
$i_{R}(t)=\frac{i(t)}{R}, t>0$
The current through the capacitor

$$
\begin{aligned}
& i_{C}(t)=C \frac{d v(t)}{d t}=C\left[-\alpha e^{-\alpha t}\left[C_{1} \cos (\beta t)-C_{2} \sin (\beta t)\right]+e^{-\alpha t}\left[-\beta C_{1} \sin (\beta t)-\beta C_{2} \cos (\beta t)\right]\right], t>0 \\
& \left.i_{C}(t)=C\left[-\alpha C_{1}-\beta C_{2}\right] e^{-\alpha t} \cos (\beta t)+C\left[\alpha C_{2}-\beta C_{1}\right] e^{-\alpha t} \sin (\beta t)\right], t>0
\end{aligned}
$$

```
    %
    tmax = 10*1/alpha; % selection of maximum t
    t = 0 : tmax/1000 : tmax;
    if alpha^2 > w^2 % overdamped
    i = Is + A1*exp(s1*t) + A2*exp(s2*t);
    C1 = L*s1*A1;
    C2 = L*s2*A2;
    v=C1*exp(s1*t)+C2*exp(s2*t); % Voltage across inductor v(t)
    iR=v/R; % current through the resistor
    ic=C*s1*C1*exp(s1*t)+C*s2*C2*exp(s2*t); % current through the capacitor
    elseif alpha^2 < w^2 %underdamped
    i = Is + exp(-alpha*t).*(B1*cos(beta*t) + B2*sin(beta*t));
    C1 = L*(-alpha*B1+beta*B2)
    C2 = L*(alpha*B2+beta*B1)
```

```
    v = C1*exp(-alpha*t).*cos(beta*t) - C2*exp(-alpha*t).*sin(beta*t);
    iR=v/R;
        % Voltage across induc
        % current through the
                                % current through the
    ic=C*(-alpha*C1-beta*C2)*exp(-alpha*t).* cos(beta*t) +C*(alpha*C2-beta*C1)*exp(-alpha*t).*s
else %Critica lly damped
    i = Is + A1*exp(s*t) + A2*t.*exp(s*t);
    C1= L*(s*A1+A2);
    C2 = L*s*A2;
    v=C1*exp(s*t)+C2*t.*exp(s*t); % Voltage across inductor v(t)
    iR=v/R;
    ic=C* (s*C1*exp(s*t)+C2* exp(s*t)+s*C2*t.*exp(s*t));
end
figure (1)
plot(t,i)
hold on;
plot(t,iR)
plot(t,ic)
grid on
xlabel('t [s]')
ylabel('i(t) [A]')
legend('i(t)', ' i_R(t)','i_C(t)')
hold off
```


figure (2)
plot(t, v)
grid on
xlabel('t [s]')
ylabel('v(t) [V]')


## References

Electric Circuits, James Kang, 1st ed. Cengage Learning, 2018

