

Computational Geology 30

Archimedes' Bath – A Better Story

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Topics this issue-

Mathematics: arithmetic involving density, volume, mass and weight; equation manipulation; unit conversions; harmonic mean.

Geology: Stokes' Law of settling; cover-collapse sinkholes; stress and strength.

INTRODUCTION

Speaking of Archimedes (CG-28, Jan. 2005, Archimedean Slices and the Isostatic Sphere), it is worth thinking about Archimedes Principle in the context of problem solving using spreadsheets. Nearly all geology majors take a year of college physics, where they encounter hydrostatics and, therefore, Archimedes Principle. In many cases, they also encounter Archimedes Principle in their hydrogeology courses, and in some cases even in rocks and minerals courses in the context of measuring density. This exposure to Archimedes Principle is a good thing, of course, but there is a problem. Presentation of the principle is almost always accompanied by the story about how Archimedes overflowed his bath and – Eureka! – how he devised a method to determine whether a particular crown was a fraud. The problem is that Archimedes could not have done it that way. The numbers do not work out. The story does not give Archimedes enough credit.

The substance and calculations for the following story about the crown are detailed in a remarkable website on Archimedes by Chris Rorres, Professor Emeritus of Mathematics at Drexel University: <http://www.mcs.drexel.edu/~crrorres/>. In addition to the page on "The Golden Crown", which motivates this essay, there are pages about "Archimedes Claw," "Archimedes Screw," "The Lever," "Archimedean Solids," "Archimedes Crater," and many more. My contribution to this essay is a couple of spreadsheets and some comments about geological applications.

THE STORY

As the story goes, Hiero II (306?-215 B.C.), king of Syracuse, commissioned the manufacture of a gold crown. He suspected that the crown he was presented was a fraud; that is, he suspected it contained some silver instead of gold. He asked Archimedes to determine whether his suspicion was correct. Archimedes, of course, was not to harm the crown in any way. He needed to perform a nondestructive test.

While bathing, Archimedes realized that when he got into the bath, he displaced water and the water overtopped the bath. This event gave Archimedes the idea that he could test the gold crown as follows: (a) place a piece of gold the same weight as the crown in a vessel of water and fill the vessel to the brim, (b) remove the gold and place the crown in the same vessel of water, and (c) notice whether the water overflowed the vessel. If the

crown was a fraud, it would be less dense and, therefore, displace more water, which would overflow the vessel.

The story is attributed to Vitruvius, a Roman architect of the first century AD.

CALCULATION 1: DISPLACEMENTS

As Dr. Rorres explains, the golden crowns of the time were wreaths (see his Website for illustrations). The largest one yet found has a maximum ring diameter of 18.5 cm and a mass of 714 g. Some of the leaves are missing. For the sake of argument, let's say (with Dr. Rorres) that Hiero's crown had a mass of 1000 g and would fit in a vessel 20 cm in diameter.

The spreadsheet of Figure 1 shows the calculation of how much the water level would rise if the 1000-g golden crown were placed in a 20-cm-diameter bowl, and how much it would rise if the crown contained some silver.

Cell C4 gives the mass of the crown. Cell C8 calculates the volume of the crown assuming it is gold (density = 19.3 g/cm³, Cell C7). The result, 51.81 cm³, is the volume of water that would be displaced by immersing the gold crown. How much would the water rise? The answer is 0.165 cm (Cell C22), and is found from the volume of displaced water (Cell C8) and the area of the vessel (Cell C19), which follows from the diameter (Cell C18).

Now, how much would the water rise, if the crown contained some silver? Let's say (with Dr. Rorres) that 30% by weight of the crown is silver (Cell C11). The calculation is done in the same way except that, now, one needs to find the density of the gold-silver alloy. If 30 wt% of the crown is silver with density 10.6 g/cm³ (Cell C12), and 70 wt% of the crown is gold with density 19.3 g/cm³ (Cells C13 and C14, respectively), what is the average density? We want the harmonic mean (CG - 21, Sept. 2002, The Harmonic Mean), which is calculated in Cell C15. The result, 64.57 cm³, is the volume of water that would be displaced by immersing the imposter crown. Using the same area (Cell C19), this would cause a rise of 0.206 cm (Cell C23).

Compare the two answers: a rise of 1.65 mm for the crown if it's truly gold and 2.06 mm for the crown if it's a fraud – a difference of less than half a millimeter!

Here is what Dr. Rorres says about the result: "This is much too small a difference to accurately observe directly or measure the overflow from considering the possible sources of error due to surface tension, water clinging to the gold upon removal, air bubbles being trapped in the lacy wreath, and so forth."

In the spreadsheet, Cells C4 (mass), C11 (percentage of silver), C18 (diameter of water vessel), C7 (density of gold) and C12 (density of silver) are all input numbers. All the other cells are formulas. It is worth changing the numbers in the first three to see the effect of the assumptions on the final result. With 1000 grams, 30% silver, and 20-cm diameter, the result is a 0.41-mm difference. What if the mass of the wreath was less, or there was less silver, or the bowl had a larger diameter?

	B	C	D	E
	ARCHIMEDES BATH -- DISPLACEMENTS			
4	Mass of Hiero's crown	1000 g	Adjustable	
5				
6	Assume crown is gold			
7	Density of gold	19.3 g/cm ³	given	
8	Volume of gold crown	51.81 cm ³	mass/density	
9				
10	Assume crown is gold/silver			
11	wt percentage of silver	30	Adjustable	
12	density of silver	10.6 g/cm ³	given	
13	wt percentage of gold	70	from C11	
14	density of gold	19.3 g/cm ³	C7	
15	density of gold/silver	15.49 g/cm ³	harmonic mean	
16	Volume of gold/silver crown	64.57 cm ³	mass/density	
17				
18	Diameter of water vessel	20 cm	Adjustable	
19	Area of water level	314.16 cm ²	from C18	
20				
21	Rise in water level			
22	Gold crown	0.165 cm	volume/area	
23	Gold/silver crown	0.206 cm	volume/area	
24	Difference	0.041 cm	C24-C23	
25	convert	0.41 mm	RESULT	

Figure 1. Spreadsheet calculating and comparing the rise in water level resulting from immersing a golden vs. an imposter crown.

ARCHIMEDES PRINCIPLE

The numbers do not work out in Vitruvius's story. Even worse for teaching Archimedes Principle is that Archimedes Principle is not involved in the story. Archimedes Principle is this:

A submerged body is buoyed up by a force equal to the weight of fluid it displaces.

Vitruvius's story makes no mention of forces.

A BETTER STORY

Vitruvius's story does not do justice to Archimedes. Quoting Rorres: "... in spite of Vitruvius's description of it as 'the result of boundless ingenuity,' the method requires much less imagination than Archimedes exhibits in his extant writings."

Dr. Rorres has a better idea - "a more imaginative and practical technique to detect the fraud." It applies Archimedes Principle. It employs levers. It would work.

The better idea involves a simple beam balance, where a pan is suspended from each end of a beam.. On one pan place the crown. On the other pan, place an equal mass of gold (a gold nugget). If you can't find a gold nugget with the same mass as the crown, then adjust the balance point (applying Archimedes' Law of the Lever, as Rorres points out). Then place the whole thing - balance, crown and nugget - in a (large) container of water. If the crown is really gold, then both it and the nugget will be buoyed up by equal forces, and they will still balance each other. If the crown contains silver, it will displace a larger weight of water, which means that it will be buoyed up by a larger force than that buoying up the nugget. As a result, the balance will tip, the nugget sinking. Isn't that ingenious? Much more so than overtopping a bath.

	B	C	D	E
	ARCHIMEDES BATH -- FORCES			
4	Mass of Hiero's crown	1000 g	Adjustable	
5	Wt of Hiero's crown	9.81 Newton	$W = mg$	
6				
7	Crown			
8	Assume crown is gold/silver			
9	wt percentage of silver	30 %	Adjustable	
10	density of silver	10.6 g/cm ³	given	
11	wt percentage of gold	70 %	from C11	
12	density of gold	19.3 g/cm ³	given	
13	density of gold/silver	15.49 g/cm ³	harmonic mean	
14	Volume of gold/silver crown	64.57 cm ³	mass/density	
15	Volume of displaced water	6.46E-05 m ³	conversion	
16	Mass of displaced water	0.0646 kg	mass*density	
17	Buoyancy force	0.63 Newtons	ARCHIMEDES	
18				
19	Nugget			
20	Mass of gold nugget	1000 g	C4	
21	Wt of gold nugget	9.81 Newton	$W = mg$	
22	Density of nugget	19.3 g/cm ³	C12	
23	Volume of nugget	51.81 cm ³	mass/density	
24	Volume of displaced water	5.18E-05 m ³	conversion	
25	Mass of displaced water	0.0518 kg	mass*density	
26	Buoyancy force	0.51 Newtons	ARCHIMEDES	

	B	C	D	E
2	Difference in force	0.12 N	C17-C26	
3	Corresponding mass	0.0122 kg	$m = W/g$	
4	convert to grams	12.2 grams	conversion	
5	Density of limestone	2.5 g/cm ³	given	
6	Volume of limestone	4.9 cm ³	mass/density	
7	Edge length of cube of ls	1.7 cm	cube root	

Figure 2. Spreadsheet calculating the buoyancy forces on a golden vs. an imposter crown

CALCULATION 2: FORCES

The spreadsheet of Figure 2A does the calculation, which follows the numbers on Dr. Rorres' website. Again we start with the mass of Hiero's crown as 1000 grams (Cell C4). This time we are interested in forces, and so we calculate that the weight of the crown is 9.81 N (Cell C5). Similarly, the mass and weight of the gold nugget are 1000 grams (Cell C20) and 9.81 N (Cell C21), respectively.

Now, we immerse both the crown and nugget into the water and calculate the volume of water displaced in each case and, from those results, the respective buoyancy forces. Block C8:C17 does the calculation for the crown assuming, again, that it is 30 wt% silver and 70 wt% gold. Block C20:C26 does the calculation for the gold nugget.

Taking the gold crown first, Block C8:C14 calculates the volume of displaced water (64.57 cm³) using the harmonic mean as we did in the spreadsheet of Figure 1. Cell C15 converts the volume to cubic meters, from which C16 calculates the mass of displaced water (remembering that the density of water is 1000 kg/m³). From the mass of the displaced water (0.0646 g), Cell C17 calculates the weight (0.63 N), which is the force buoying up the crown.

For the nugget, Cell C23 calculates the volume as 51.81 cm³ from the mass (C20) and density (Cell C22) as in the spreadsheet in Figure 1. Converting to cubic meters (C24) and finding the mass of displaced water (C25)

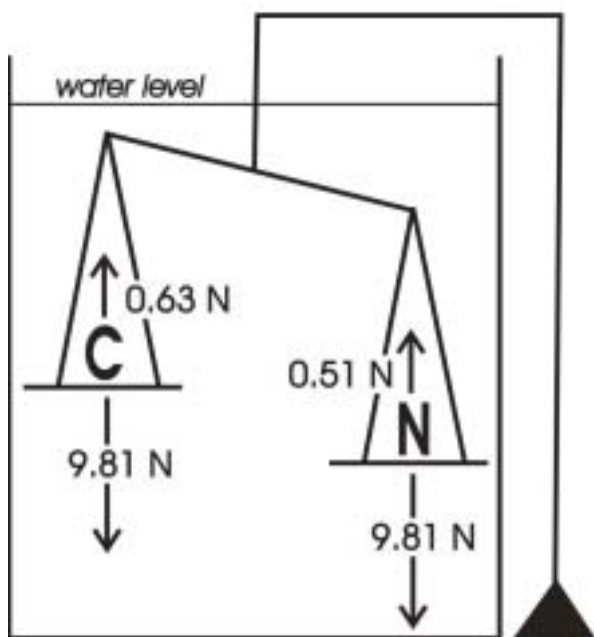


Figure 3. Summary of gravity and buoyancy forces for a setup that would solve King Hiero's problem.

leads to the weight of displaced water (0.51 N), which is the buoyancy force acting on the nugget.

The result of the calculation is shown in Figure 3. Immersed in water, the downward force on the nugget is 9.81 N and the upward force is 0.51 N for a net downward force of 9.30 N. Immersed in water, the downward force on the imposter crown is 9.81 kg, and the upward force on it is 0.63 N, for a net downward force of 9.18 N. The nugget sinks because of an imbalance of 0.12 N.

The spreadsheet in Figure 2B gets at the question, How large is 0.12 N? The answer is that a mass of 0.0122 kg (Cell C3) weighs 0.12 N. Converting to grams (Cell C4), and using a density of limestone as 2.5 g/cm³ (Cell C5), a volume of limestone of 4.9 cm³ (Cell C6) weighs 0.12 N. If this volume of limestone were a cube it would measure 1.7 cm on a side (Cell C7). So, Archimedes balance scale would be able to discern the difference if it were capable of registering a 1.7-cm cube of limestone. Quoting Rorres website again: "Scales from Archimedes time could easily detect such an imbalance in mass. Additionally, sources of error arising with Vitruvius's method (surface tension and clinging water) would not arise with this scale method."

COMMENTS

The tale spun by Vitruvius some two thousand years ago, I believe, misdirects students in their thinking about Archimedes Principle. What "principle" comes to mind when one thinks of Archimedes overtopping his bath? A submerged body displaces its own volume of water? While true, that "principle" should not have required one of history's greatest intellects to figure out. To associate that truism with Archimedes Principle is akin to saying Galileo discovered that things fall down or that Newton discovered that apples fall from trees.

What about the student who says, "Archimedes Principle: A submerged body displaces its own weight of water"? I bet that student is thinking of Vitruvius's story

and also remembering that Archimedes Principle has something to do with weight (force). But the student is incorrect. Not only is the statement not Archimedes Principle, but the statement itself is false, not a principle at all.

Perhaps the student is thinking "Archimedes Principle: A *floating* body displaces its own weight of water" (emphasis added to show the difference from the preceding statement). This statement about a floating object is true, but it is not Archimedes Principle. Archimedes Principle is more general. This statement is a special case where the force of buoyancy balances the weight of the object.

These mistakes would not happen if students think of Dr. Rorres's story rather than Vitruvius's, for then the students would be thinking of buoyancy forces and the balance or imbalance of forces. This thinking then would help in all sorts of geology courses. Let me give two examples.

Example 1: Stokes' Law

The first example is Stokes' Law, a staple of sedimentology courses. For a sphere falling through a column of fluid:

$$v = \frac{(\rho_s - \rho_f)g}{18\mu} D^2 \quad (1)$$

where v is the settling velocity; ρ_s and ρ_f are the densities of the solid and fluid, respectively; μ is the viscosity of the fluid; and D is the diameter of the sphere. Where does the $\rho_s - \rho_f$ term come from? It is Archimedes Principle coming into play in the net force propelling the sphere downward.

The net downward force is the difference between the weight of the sphere (downward force due to gravity) and the sphere's buoyancy (upward force). This difference is the driving force and is found from

$$F_{\text{Driving}} = \frac{\pi \rho_s g D^3}{6} - \frac{\pi \rho_f g D^3}{6} \quad (2)$$

The first term on the right is the weight of the sphere (specific weight, $\rho_s g$, times the volume of a sphere written in terms of diameter instead of radius). The second term on the right is the weight of the displaced sphere of fluid, namely Archimedes Principle for the buoyancy force. Equation 2 reduces to:

$$F_{\text{Driving}} = \frac{\pi(\rho_s - \rho_f)gD^3}{6\mu}, \quad (3)$$

which can be usefully compared to Equation 1.

The settling (terminal) velocity occurs when the driving force is balanced by the resisting force, and the sphere is not accelerating. Thus, for the settling velocity:

$$F_{\text{Drag}} = F_{\text{Driving}} \quad (4)$$

where F_{Drag} is the resisting force. Now, substitute Equation 3 into Equation 4, divide by Equation 1, and rearrange:

	B	C	D	E	F
2	ROCK COLUMNS SUPPORTING A ROOF				
3					
4	Block to hold up				
5	Dimensions				
6		height		12 m	
7		width1		7 m	
8		width2		8 m	
9		Effective Density		1.7 g/cm ³	
10	Columns supporting roof				
11	Dimensions				
12		width1		0.5 m	
13		width2		0.5 m	
14		Number		1	
15	→	Strength		60 MPa	
16					
17	Environment	g =		9.81 m/sec ²	
18					
19	Mass of block			1142400 kg	
20	Weight of block			11206944 N	
21					
22	Weight on each column			11206944 N	
23	Stress on each column			44827776 Pa	
24	→	convert		44.8 MPa	
25					
26	Factor of safety			1	
27					
28	Will the columns hold the load?			Yes	

	B	C	D	E	F
2	ROCK COLUMNS SUPPORTING A ROOF				
3					
4	Block to hold up				
5	Dimensions				
6		height		12 m	
7		width1		7 m	
8		width2		8 m	
9		Effective Density		2.7 g/cm ³	
10	Columns supporting roof				
11	Dimensions				
12		width1		0.5 m	
13		width2		0.5 m	
14		Number		1	
15		Strength		60 MPa	
16					
17	Environment	g =		9.81 m/sec ²	
18					
19	Mass of block			1814400 kg	
20	Weight of block			17799264 N	
21					
22	Weight on each column			17799264 N	
23	Stress on each column			71197056 Pa	
24		convert		71.2 MPa	
25					
26	Factor of safety			1	
27					
28	Will the columns hold the load?			NO	

Figure 4. Spreadsheet assessing whether a column will hold up a rectangular block.

$$F_{\text{Drag}} = 3\pi Dv \quad (5)$$

Equation 5 is the drag force on a sphere. It is known as Stokes' Law of Resistance, and, historically, it was Stokes' path to Equation 1. That is, his breakthrough was figuring out Equation 5. See Blatt et al. (1980, p. 59-66) for the physics of Stokes' Law. See Hsu, (1989, Chap. 3) for the relevant physics plus some reflections about teaching it to geology students.

Example 2: Sinkholes

Tampa is in sinkhole country. The kind of karst in west-central Florida is called "mantled karst" or "covered karst." The name means that the limestone does not crop out directly. Rather, the Eocene and Oligocene limestones of the area are covered by younger sediments, particularly a layer of Plio-Pleistocene sand. In many places, there is a Miocene clayey layer between the limestone and the sand, and in places the sandy sediments themselves are intermixed with clay layers at the base of the sand section. The clay-rich zone acts as a confining unit between the monumentally important Floridan aquifer system (carbonate rocks) and the sandy surficial aquifer system.

Sinkholes in the Tampa area reflect transport of the sand downward through the clayey beds and into cavernous zones at the top of the underlying limestone. Thus the confining unit is perforated by pipe-like structures filled with sand and organic matter from the surface layer. The sinkholes are a surface manifestation

of these pipes, where sand is moving down somewhat like it does in an hourglass. We are living on a plain at the top of a blanket of hourglass structures. Where the hourglass flow gets stuck, because of internal resistance of the clayey sands in the pipe, we have so-called cover-collapse sinkholes, and the sinkhole subsidence is episodic. Where the hourglass flow is of cohesionless sand, we have so-called cover-subsidence sinkholes. The cover-collapse sinkholes have impact because the time intervals between episodes are commonly long enough for urban development.

Numerous geomorphology and environmental science courses at USF include segments about our sinkholes. The USF GeoPark (<http://uweb.cas.usf.edu/~vacher/USFGeoPark/GeoPark.htm>) features a sinkhole. From living here we all know that water levels play a role in sinkhole episodes. For example, when it freezes (thankfully rare events lasting at most a few hours after midnight), farmers in the rural outskirts pump large amounts of water from the Floridan aquifer to spray their plants (e.g., orange trees). The resulting ice coating prevents the plants from still lower temperatures. Sinkholes follow.

The story of this sinkhole inducement goes as follows. The water table is in the surficial sands. The potentiometric surface is typically a little lower than the water table. Thus there is a hydraulic drive for downward flow through the confining bed – meaning through the sand pipes below the sinkholes. The sudden, large-scale withdrawals lower the potentiometric surface, which increases the hydraulic drive, which increases the flow through the sand pipe, which induces the sinkhole to clear its throat. "Clearing the throat" means that the pipe subsides, which means a

cover-collapse sinkhole. To the extent that the water table is also lowered, there is the added effect that the sand layer weighs more as it becomes dewatered. This increase in load acts to push down the plug in the pipe. This effect is likely to dominate where the confining bed is thin and the two aquifers are hydraulically connected. For a full account of our sinkholes see Tihansky (1999).

The point to be made here is that the second effect – the effect of dewatering – is an application of Archimedes Principle. While the material is submerged, it is buoyed up by the water, and therefore its effective weight is less. To see the magnitude of load increase during dewatering, consider the spreadsheet of Figure 4. This figure is a slight modification of a spreadsheet I use for other purposes in my Computational Geology course. I also include it in a spreadsheet module (Number 1.3 within a set on the density of the Earth and Earth materials) at a quantitative literacy website at http://www.evergreen.edu/washcenter/modules/star_t.htm.

The spreadsheet in Figure 4 gives a calculation to determine whether a column of rock will hold up a particular portion of the roof. The problem fits with the workbook we use for the class (Derrington, 1998), which unfortunately is out of print. As this problem is set up, there is one column (Cell E14) supporting a 7 m x 8 m section of the roof (Cells E7 and E8) that is 12 m thick (Cell E6). The column is 0.5 m x 0.5 m in cross-section (Cells E12 and E13). Will the column hold? If the stress (weight per area) on the column (Cell E24) is less than the compressive strength of the rock in the column (60 MPa; Cell E15), the logic statement in Cell E28 will answer "Yes."

The key line for the point about Archimedes Principle is Row 9, the effective density. The value entered in Cell E9 is 1.7 g/cm³. This is $\rho_s - \rho_f$ as in Stokes' Law: the density of a 2.7-g/cm³ rock reduced by the density of the water (1.0 g/cm³) it displaces. Thus the spreadsheet answers the question for a submerged case.

Lower the water level so that the roof rock is no longer buoyed by the water, and the density increases to 2.7 g/cm³ – an increase of 37%. Each lb (force) of submerged weight becomes 1.37 lb when it is stranded high and dry. Change Cell E9 to 2.7; then, Cell E24 becomes 71.2, and Cell E28 becomes "No."

CONCLUDING REMARKS

Archimedes Principle is one of history's great physical concepts. Vitruvius's story, however, under-values both the richness of the concept and the insight of the genius who figured it out. The subject is worthy of care in

geology courses that apply physical principles to geologic processes. Using Rorres's solution to Hiero's question – rather than Vitruvius's – should improve student understanding of Archimedes Principle and prepare them for using it outside the realm of introductory physics.

Thinking of forces and buoyancy should also help with two classic Archimedes questions:

- Consider ice cubes in a glass of water. If the ice melts, will the water rise? (Geologically, if the floating Arctic Ice melts, will sea level rise?)
- Consider a geologist in a boat in a small pond. The geologist has collected a mighty pile of rocks. The geologist's assistant doesn't like the thought of carrying the rocks to the truck and throws them overboard. Does the level of the pond rise (theoretically)?

The second question deserves some context. Beth Fratesi and Don Seale, two graduate students at USF, posed a version of the problem to me. They are alumni of Mississippi State University, where they got the question as a homework assignment from John Mylroie. So I called John. Turns out that he got the question on his PhD oral exam some thirty years ago.

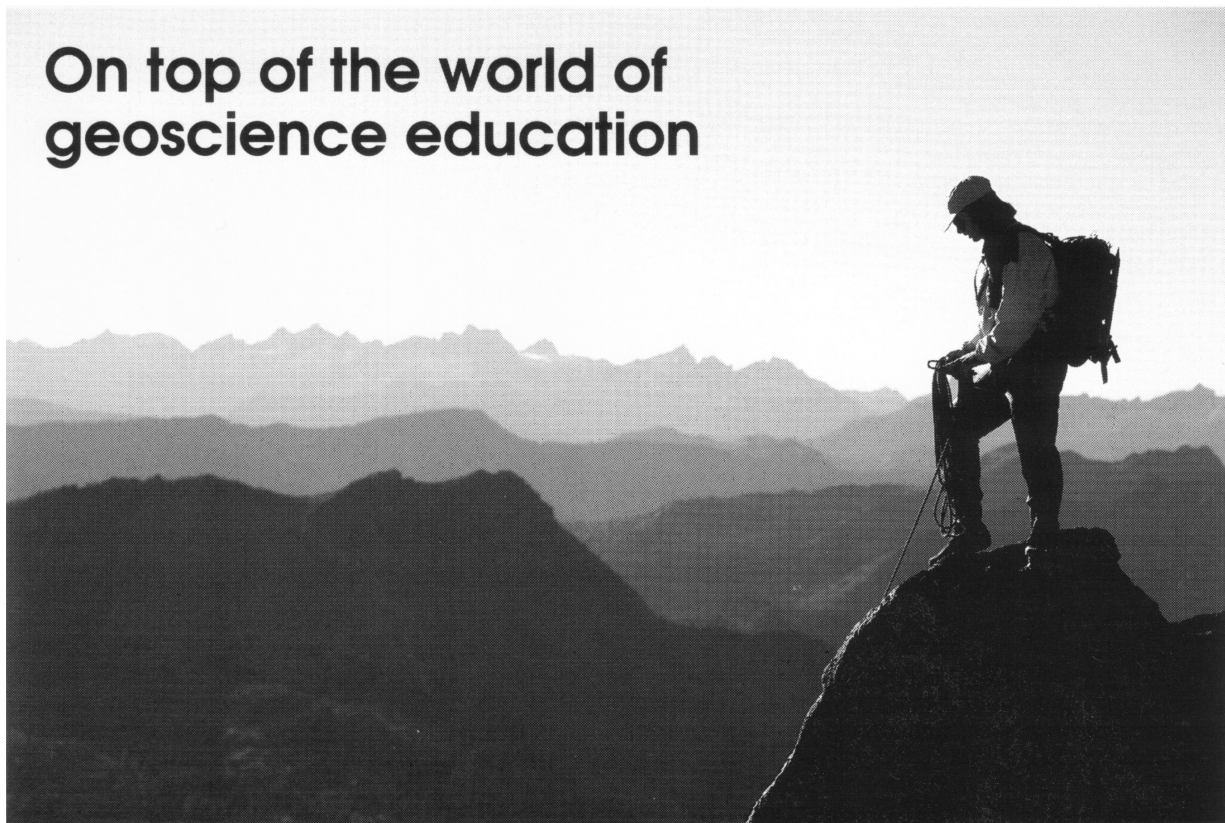
The story shows how Archimedes Principle problems are handed down from generation to generation. This is a good one. I hope it continues.

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Note to readers of Computational Geology: Column 29, printed in the March, 2005 issue suffered from a post-production glitch in font translation, the resulting column ran with no Greek letters. Therefore, the column has been rerun in this issue. Thank you for your patience and understanding.

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