Part of a lab handout posted on Blackboard:

Exercise 2 (special thanks to Giorgio Proestos and Allan Kuan of CIV102F for helping put this exercise together)

One of the main issues in the design of structures for dynamic loads, such as earthquakes or wind forces, is to establish a good estimate of the natural period or periods of vibration. In earthquake engineering, the first fundamental mode of vibration is used to estimate the acceleration demand, and in turn, the force demand on a structure. The structure acts like a filter and filters the ground motion signal until it reaches the masses. The dynamic response of the system depends on its physical characteristics, e.g. its mass and stiffness. Elongating the period (reducing the natural period of the structure) is a good way to avoid damage from earthquakes, since the frequency content of earthquake is quite high, and this is accomplished by modifying the design of the structure.

Typically the number of storeys a building has will dictate an approximate number of modes of vibration the structure will be excited in. The higher modes require far too much energy to excite them and usually only the first 2 to 5 modes (or perhaps 10 if the structure is very tall) are important.

For a two storey structure the problem condenses to two degrees of freedom described by a set of coupled, second order differential equations. The derivation of this model comes directly from free body diagrams of the masses. A two-storey building can be modelled as seen in the figure below, with the mass concentrated on each floor, and the stiffness concentrated on end columns.

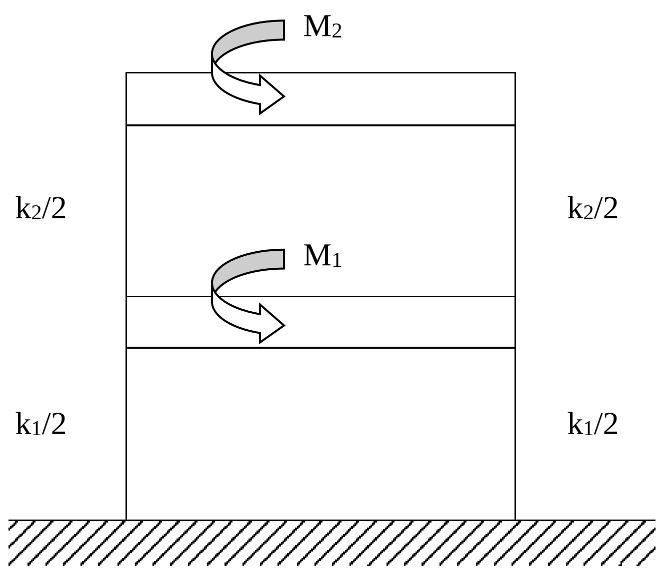
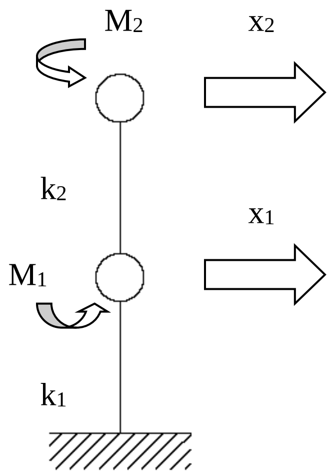


Figure 1 - Model of a two-storey building

By lumping the stiffness on each floor, the model can be further simplified into a system of two masses coupled by springs, or what is commonly referred to as the “Lollipop Model”.



Four forces are considered when taking a free body diagram of each mass:

- An inertial force from the acceleration of the mass
- Damping forces from the viscous damping associated with velocity
- Spring forces as described by Hooke’s law
- A time-varying externally applied force

In this exercise, you are asked to simulate the building’s behaviour even when an externally applied force is assumed not to be present, i.e. under what is called free vibration. We will also choose to ignore the viscous damping forces. Under these assumptions, the mathematical model of the building to be simulated consists of the following two, coupled, second order differential equations:

$$m_1\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$
$$k_1 = k_2 = 4.66\ \text{kN/mm}$$
$$m_1 = 0.0917\ \text{kNsec}^2/\text{mm}$$
$$m_2 = 0.0765\ \text{kNsec}^2/\text{mm}$$

Using initial velocities of zero and $x_1(0) = 100$ mm and $x_2(0) = 50$ mm, numerically solve this IVP using the improved Euler method for the horizontal displacements of the first and second storeys, i.e. $x_1(t)$ and $x_2(t)$.

Develop a figure showing the resulting numerical solutions for $t \in [0, 10\,\text{sec}]$, using different line types for different time steps *N*. Use the subplot command to plot $x_1(t)$ versus *t* on the top plot and $x_2(t)$ versus *t* on the bottom plot. Label the axes and add a legend showing the time step *N* associated with each line type. By studying the approximate solutions using different numbers of time steps, how many time steps are needed in order to obtain a reasonably accurate solution?

The final part of the exercise is to perform some analysis on the simulation results and relate these results to our earlier work on matrices and their eigenvalues.

By examining your simulation results, observe the resonant frequencies. Try to estimate these resonant frequencies from the data by estimating their periods and converting this to an angular frequency (radians/sec).

Next, open the Graphical User Interface (GUI) for the System Identification Toolbox in Matlab by typing 'ident'. Select the drop down menu for importing data and select Time Domain Data. Under the Workspace Variables, enter the name of the vector containing $x_1(t)$ as the Output, set the Starting Time to zero, and set the Sample Time to the time step Δt that goes with $x_1(t)$. Click "Yes" when the next window appears asking if a time series should be created from the output variable. When you do this, the data will appear is the Data Views. If you check the Time plot, you will see a time domain plot of the data. When you check the Data spectra, you will see a periodogram of the data which is the absolute square of the Fourier transform of the data. This is a frequency domain plot that clearly shows the resonant frequencies contained in $x_1(t)$. Determine the frequencies at which the system is resonating. It will help if you add a grid to the plot from the Style menu and recognize that the periodogram is a log-log plot. Repeat by examining the resonant frequencies associated with $x_2(t)$.

Now for the pièce de résistance (French for the most important or remarkable feature). Determine the eigenvalues of the system matrix using the built in function in Matlab called 'eig'. You will see that the eigenvalues come in complex conjugate pairs and correspond to the resonant frequencies or natural frequencies of the system. Voilà. Lots more to come when you take MAT292F – Calculus III in second year.

Follow-up announcements posted on Blackboard:

Exercise 2 Lab 2 results

I have posted on Bb two plots showing the results from Exercise 2 Lab 2. One plot shows the time domain results for x1 and x2 (the horizontal displacements of m1 and m2, respectively) and the other plot shows these results in the frequency domain via the periodogram.

A physical model of a 3 degree of freedom structure representing a 3 storey building being forced sinusoidally at its base may be found at

https://www.youtube.com/watch?v=OaXSmPgl1os .

This video was produced by Professor Kwon in Civil Engineering.

In the video you will see that the period of the sine wave is gradually decreased to excite the structure in its first mode and then in its second mode. There should be a third mode in this case because there are three masses, but the third mode was probably too difficult to excite on that table.

More on Exercise 2 Lab 2

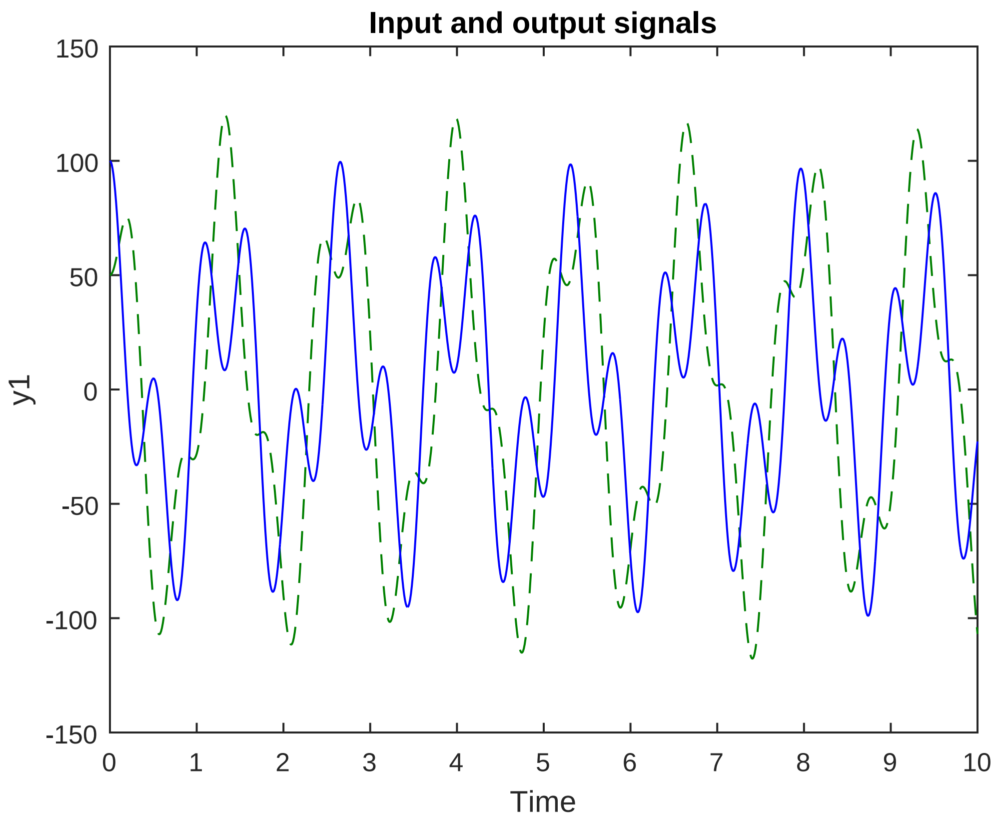
I want to refer you back to the simulation results I posted on Bb for the Lollipop model of the two storey structure and the YouTube video showing a physical model for a three storey structure. The state matrix for the Lollipop model, taking Y=[x1;x2;x1';x2'], is given by:

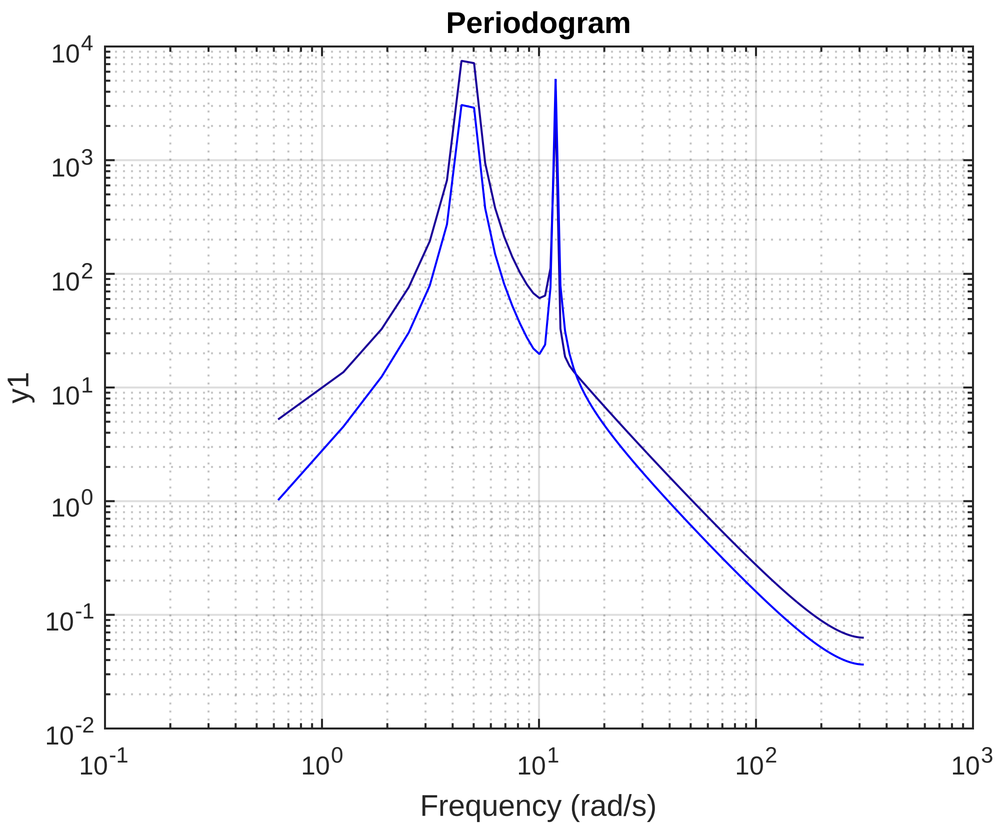
A=[0 0 1 0;0 0 0 1;-101.6358 50.8179 0 0;60.9150 -60.9150 0 0]

and using the 'eig' function in Matlab, the eigenvalues of A are +/-4.69i and +/-11.85i where i=sqrt(-1).

The eigenvalues of A determine the dynamic behaviour of this system. The fact that the eigenvalues have complex parts associated with i=sqrt(-1) means that the system responds in an oscillatory manner. The fact that the eigenvalues have no real parts means that the oscillations induced by the nonzero initial conditions do not increase in magnitude but also do not decrease in magnitude. Finally, if you look at the periodogram of the time domain responses x1 and x2, the oscillations contain two resonant frequencies and these frequencies are predicted by the complex eigenvalues (expressed in radians/sec).

Two figures posted on Blackboard referred to in the first announcement


Input and output signals
150
100
50
0
-50
-100
-150
y1
0
1
2
3
4
5
6
7
8
9
10
Time


Periodogram
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
y1
10^{-1}
10^0
10^1
10^2
10^3
Frequency (rad/s)